Start of Lecture 16

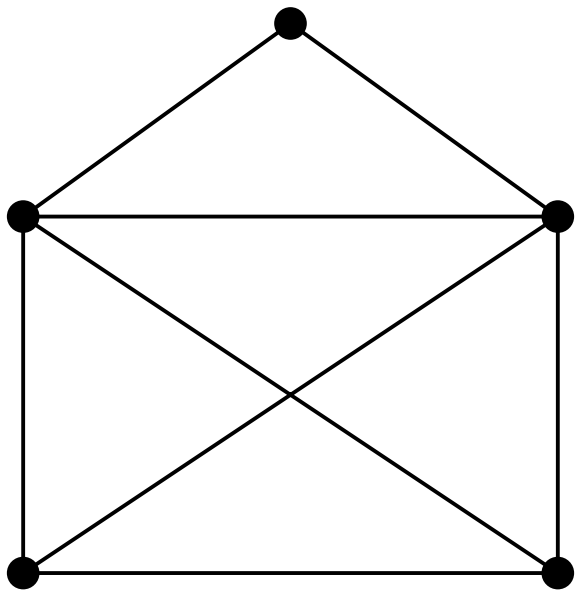
Last time: Paths and Cycles

Today: Eulerian Paths, Isomorphism, Variants



Eulerian Paths: Example

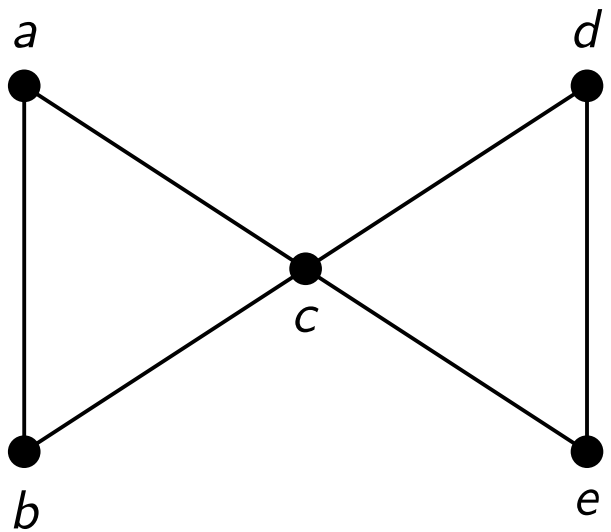
Example: Can you draw this graph without lifting your pencil and without drawing any edge twice?





Eulerian Paths and Cycles

Definition: A path or cycle in a graph is Eulerian if it uses every edge of the graph exactly once.

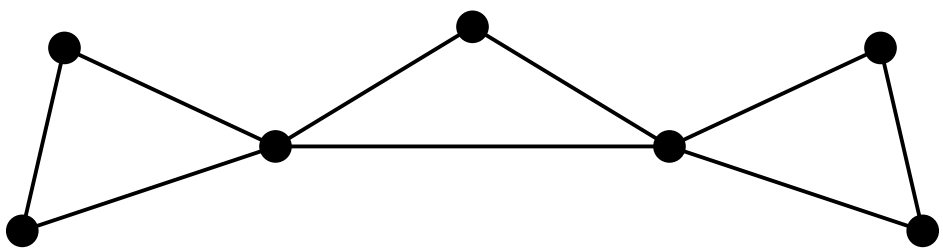


Eulerian cycle: c, a, b, c, d, e, c.

Eulerian Cycles



Theorem: A connected graph with at least one edge has an Eulerian cycle if and only if every vertex has even degree.

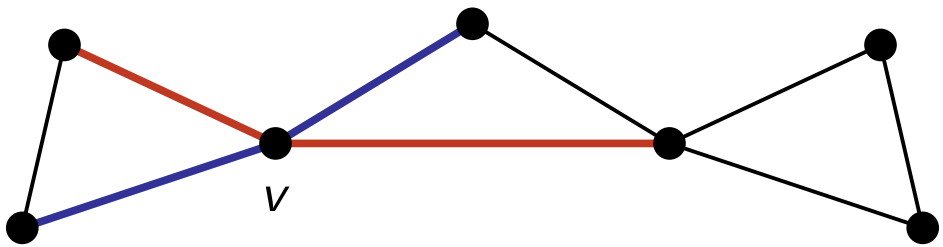


Eulerian Cycles



Theorem: A connected graph with at least one edge has an Eulerian cycle if and only if every vertex has even degree.

Proof Idea: ($\Rightarrow$) Every arrival pairs with a departure, including at the start/end vertex. Thus every degree is even.



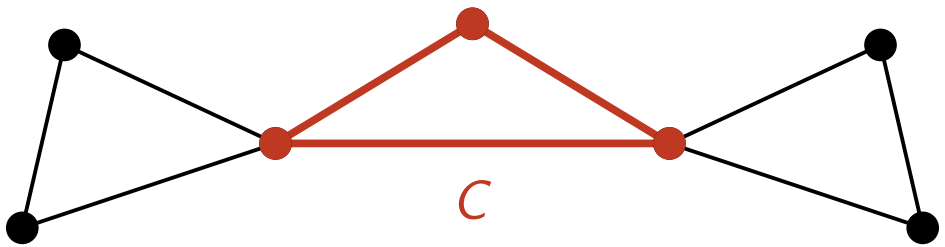
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($\Leftarrow$) By strong induction on $|E|$. Every degree is at least 2, so there is a simple cycle C.



Eulerian Cycles

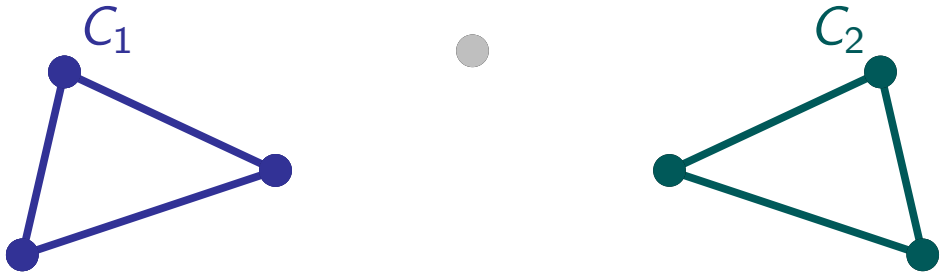


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Delete the edges of C. Degrees stay even. By induction, each remaining component with edges has an Eulerian cycle.



Eulerian Cycles



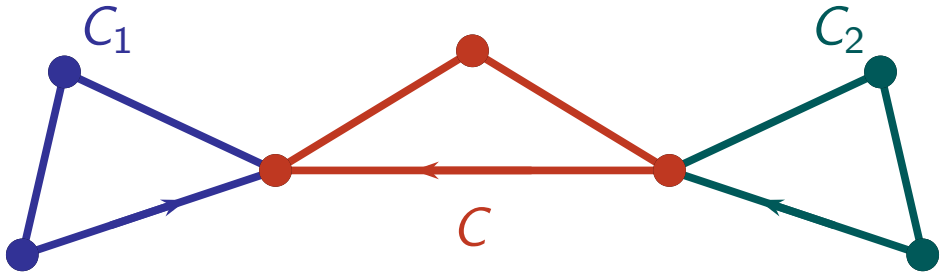
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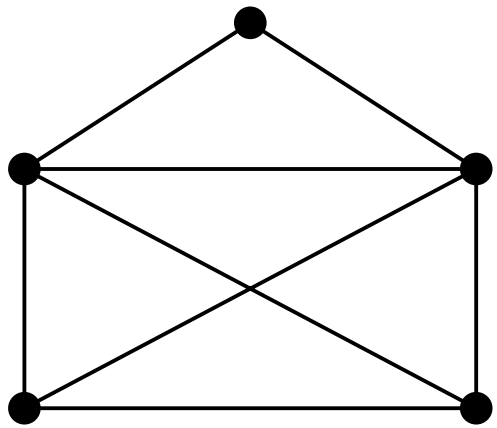
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Follow C, detouring around each component's Eulerian cycle. Every component meets C by connectedness, so every edge is used exactly once.



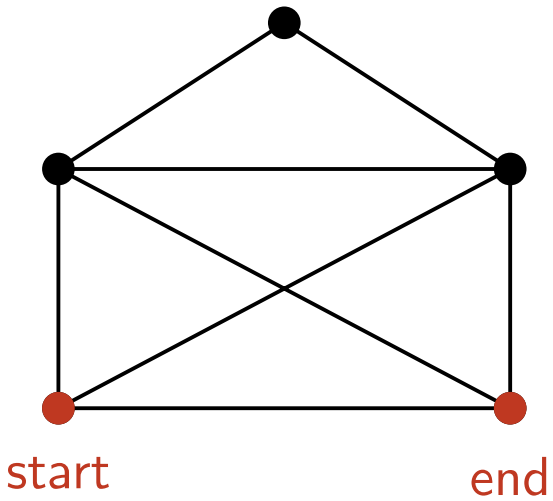
Eulerian Paths

Theorem: A connected graph has an Eulerian path if and only if it has exactly 0 or 2 vertices of odd degree.



Eulerian Paths

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Proof Idea: Same idea as Eulerian cycles proof but 2 odd degree vertices can be start and end of path.

Eulerian Cycles: Dominoes

Example: A domino set has $\binom{7}{2}$ dominoes, one for each pair of distinct numbers from 0 to 6.

Can you lay all of them out in a closed ring so that touching ends match?



Eulerian Cycles: Dominoes

Claim: The dominoes can be arranged in a closed ring so that all touching ends match.



Eulerian Cycles: Dominoes

Claim: The dominoes can be arranged in a closed ring so that all touching ends match.

Proof Idea: Let each number from 0 to 6 be a vertex, and each domino be an edge joining its two numbers.

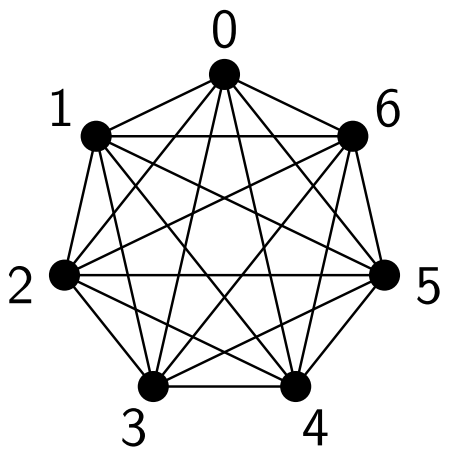


Eulerian Cycles: Dominoes

Claim: The dominoes can be arranged in a closed ring so that all touching ends match.

Proof Idea: Let each number from 0 to 6 be a vertex, and each domino be an edge joining its two numbers.

This gives K_7, which is connected and has degree 6 at every vertex. Since every degree is even, it has an Eulerian cycle.







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Eulerian Paths: Stitching

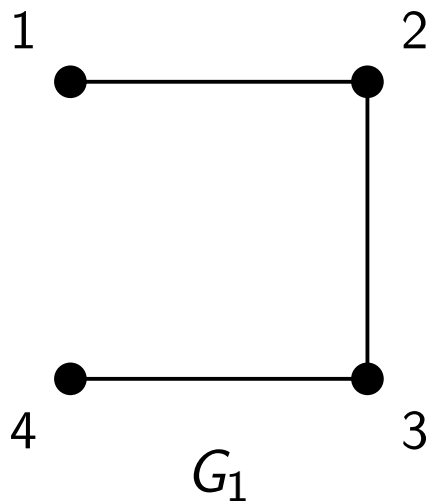
I use this sometimes to determine how to make decorative stitching patterns:



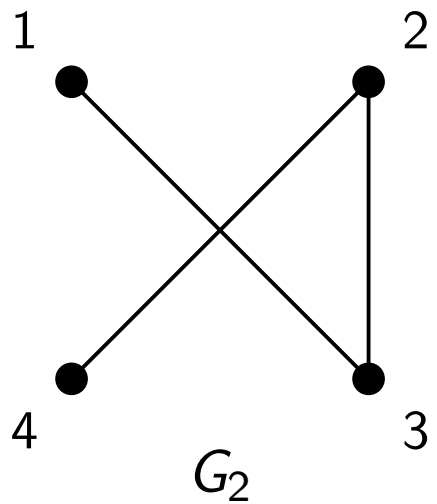


Graph Isomorphism

Example: Are these two the same graph?



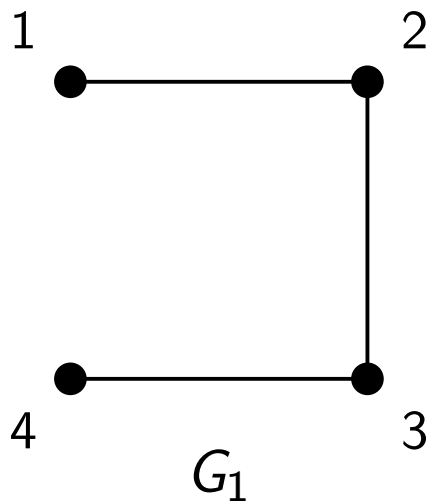
G_1



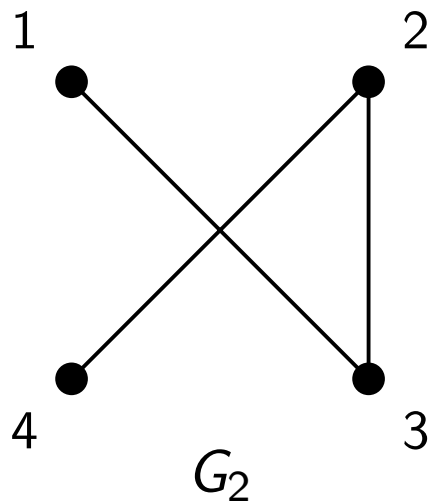
G_2

Graph Isomorphism

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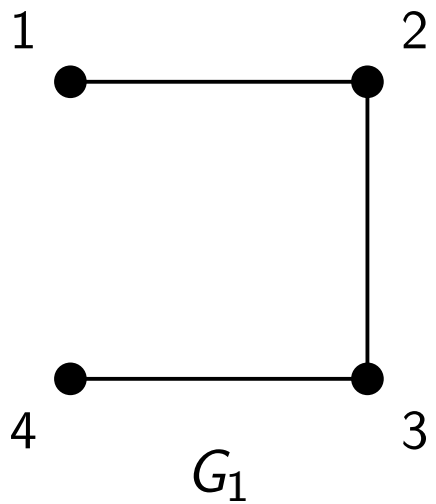


G_2

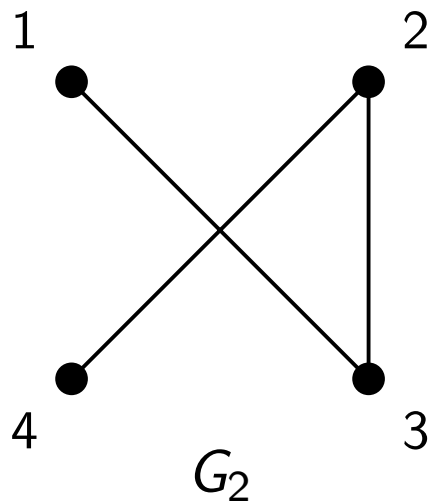
No. They have the same vertex set but different edge sets.

Graph Isomorphism

Example: Are these two the same graph?



G_1



G_2

No. They have the same vertex set but different edge sets.

But they have the same "shape": both are simple paths on four vertices. Only the vertex labels differ.

Graph Isomorphism

Definition: Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \to V_2$ such that, for all $a, b \in V_1$,

$$\{a, b\} \in E_1 \quad \iff \quad \{f(a), f(b)\} \in E_2.$$



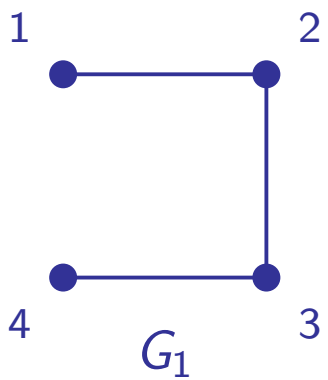
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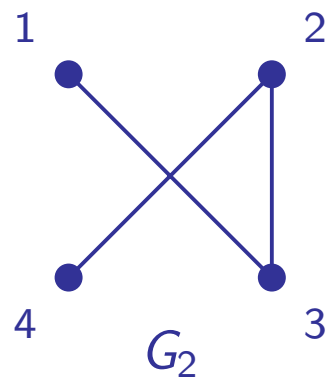
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$$\{a, b\} \in E_1 \quad \Longleftrightarrow \quad \{f(a), f(b)\} \in E_2.$$

Example: In the previous example, an isomorphism is

v	1	2	3	4
$f(v)$	1	3	2	4

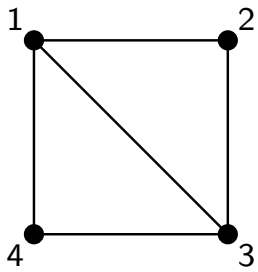


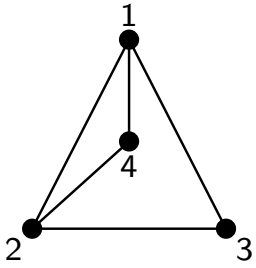


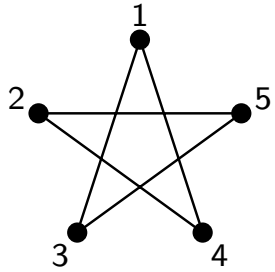


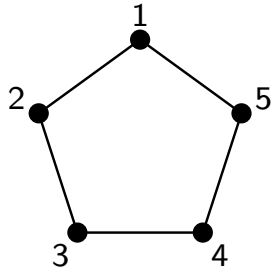
Graph Isomorphism: Examples

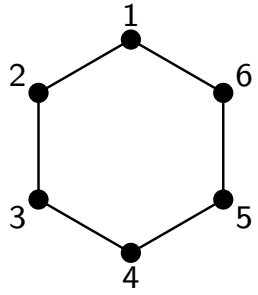
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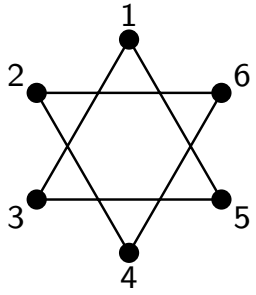


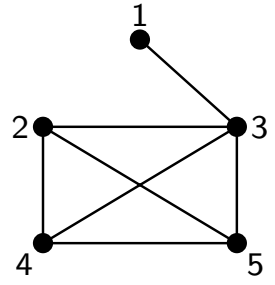


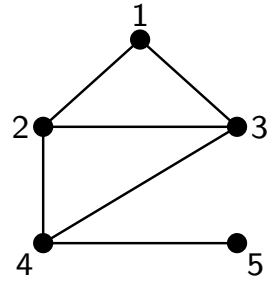


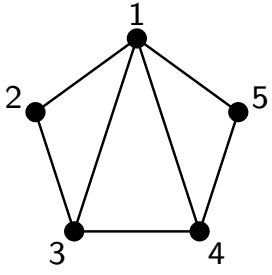


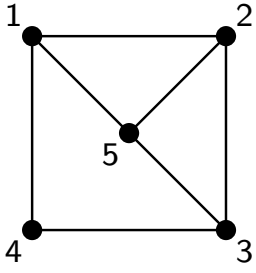


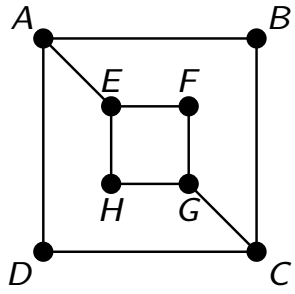


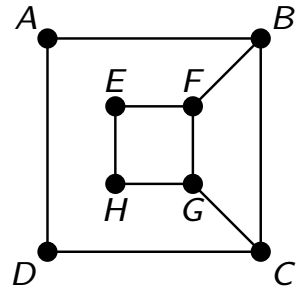






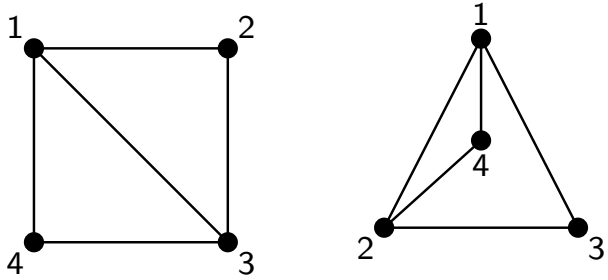


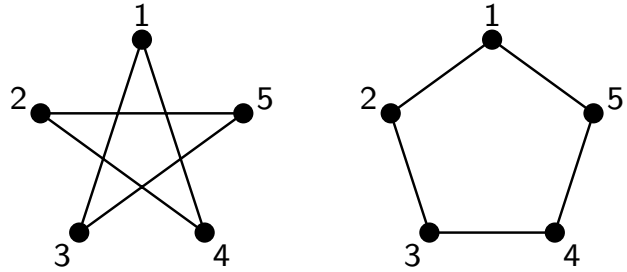




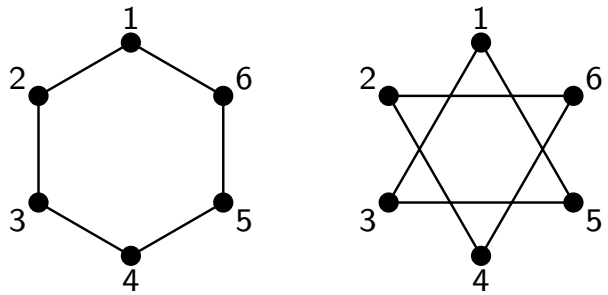
Graph Isomorphism: Examples

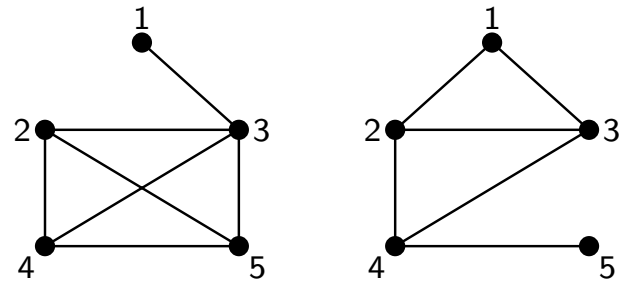
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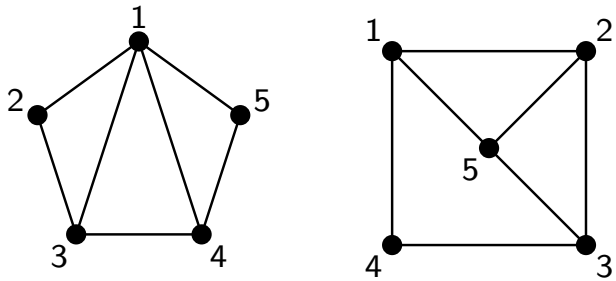


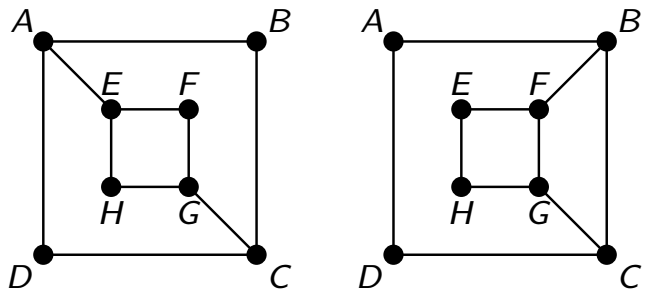


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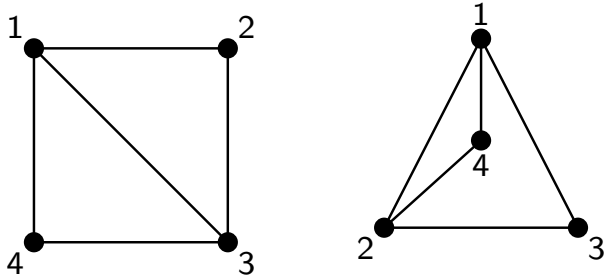




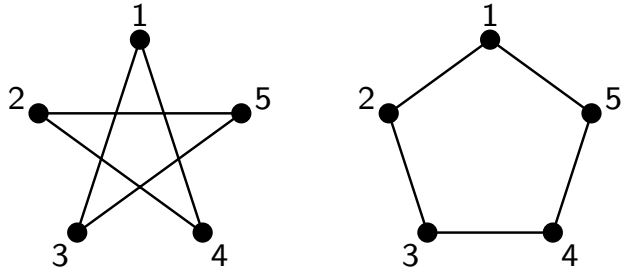


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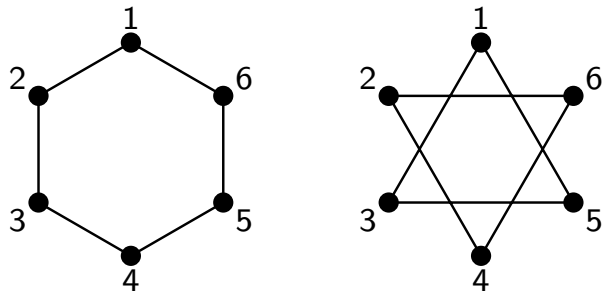
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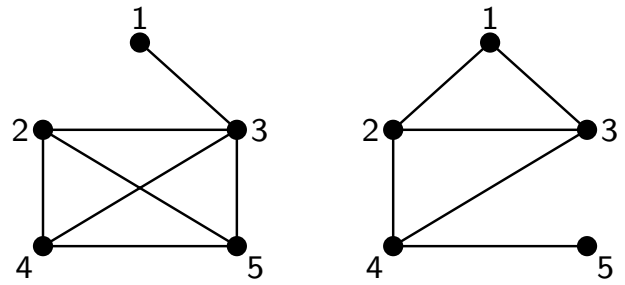


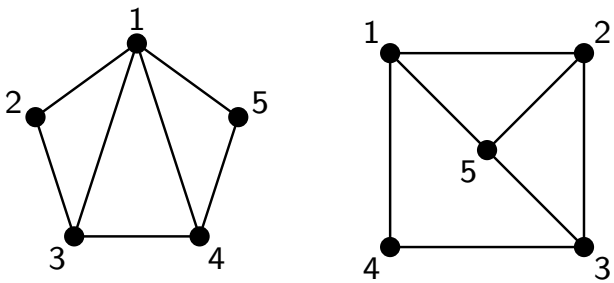
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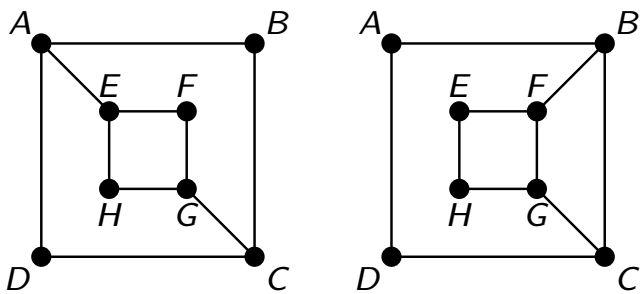


Yes: both are 5-cycles.



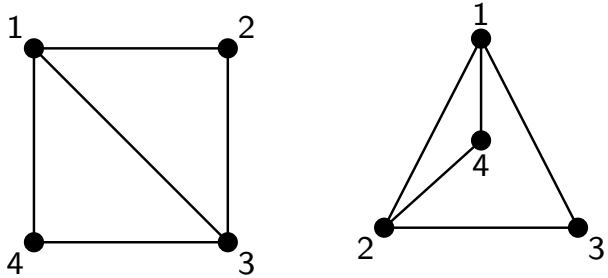




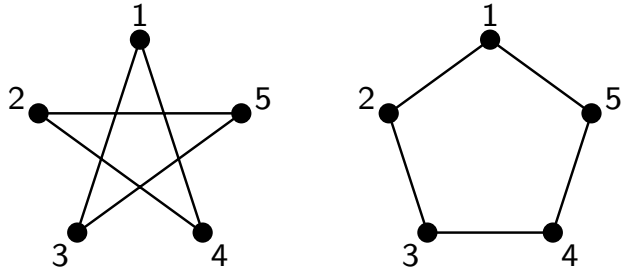


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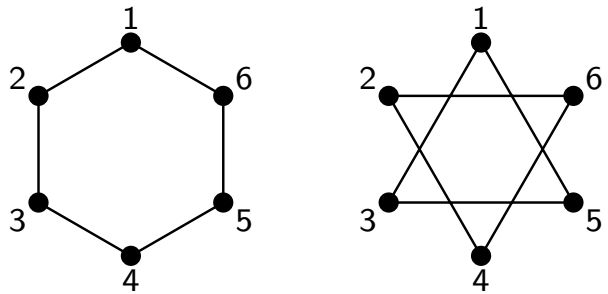
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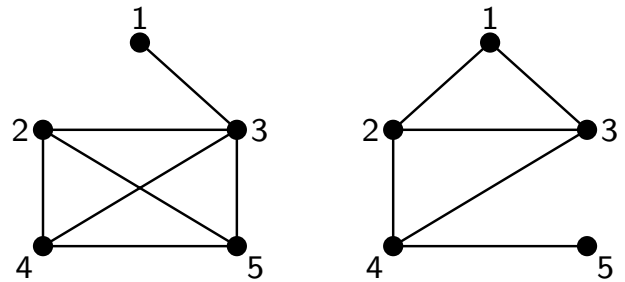
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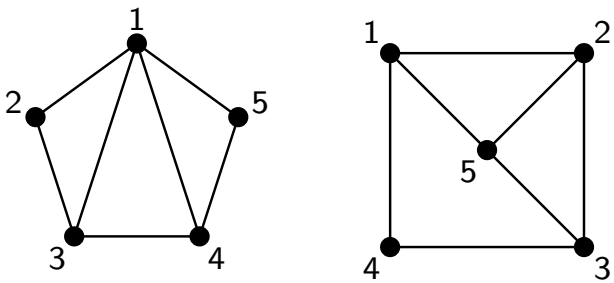


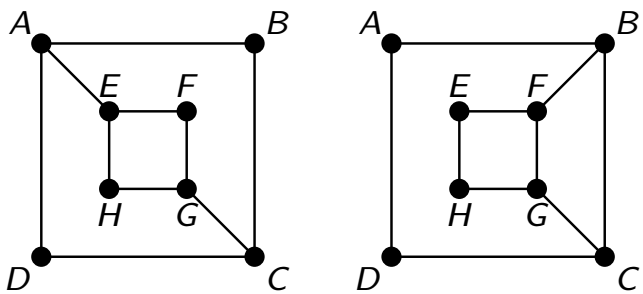
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No: 1 vs. 2 components.

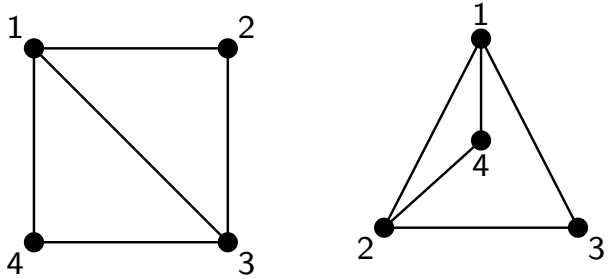




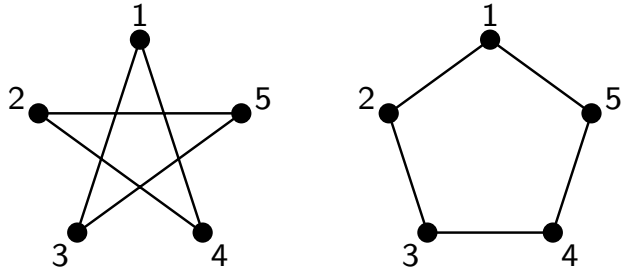


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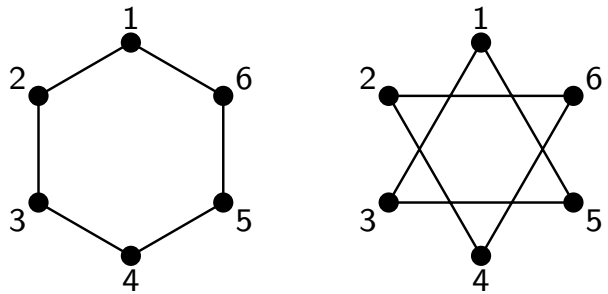
For each pair, are they isomorphic?



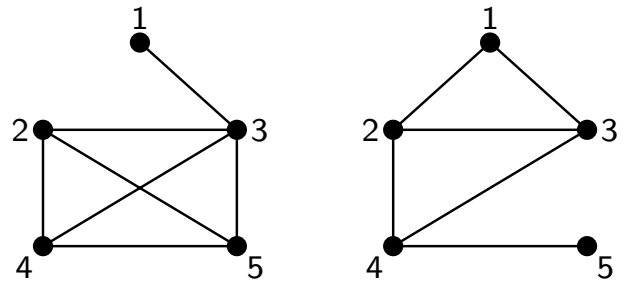
Yes: swap 2 and 3.



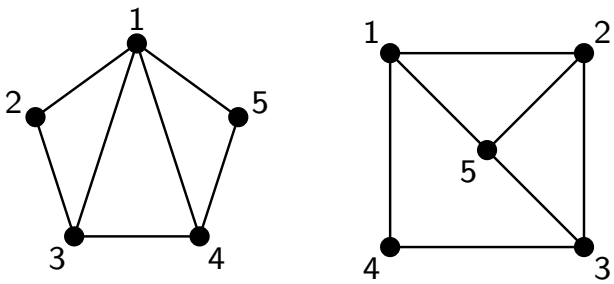
Yes: both are 5-cycles.

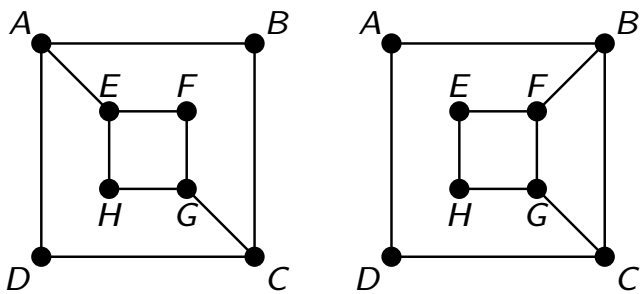


No: 1 vs. 2 components.



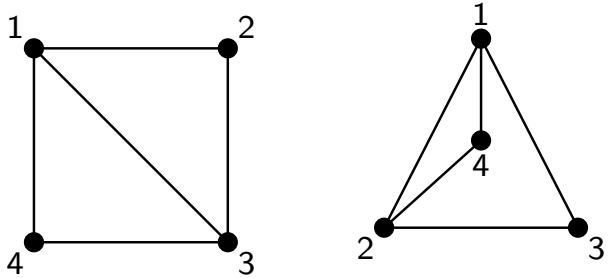
No: 7 vs. 6 edges.



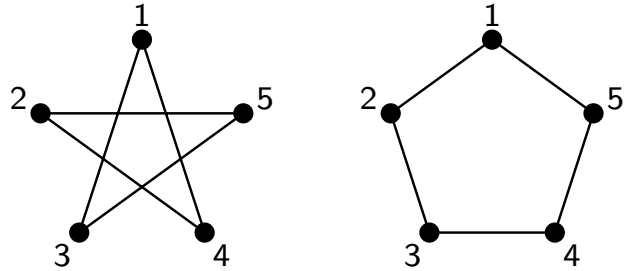


Graph Isomorphism: Examples

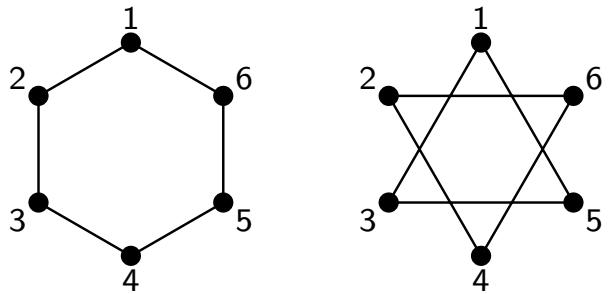
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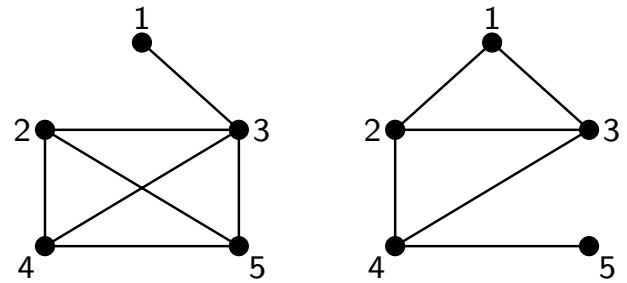
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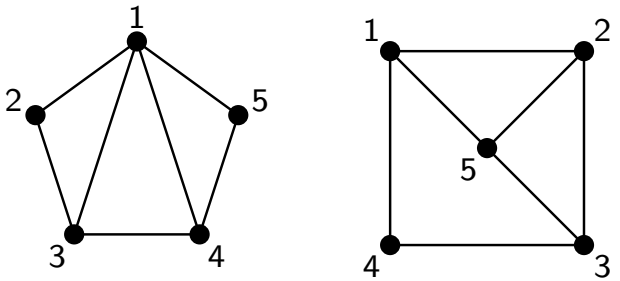
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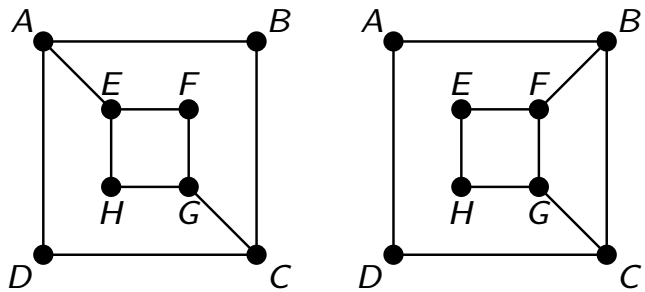
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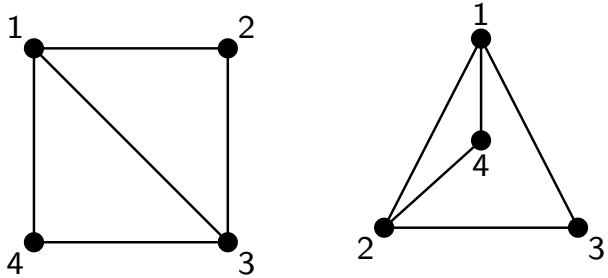


No: maximum degrees 4 and 3.

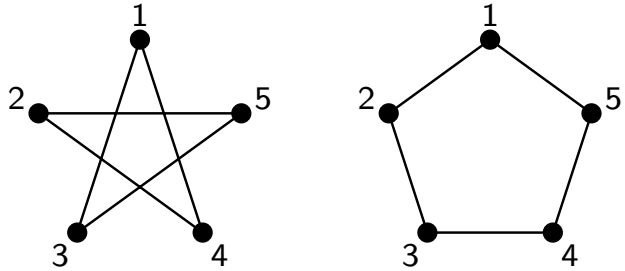


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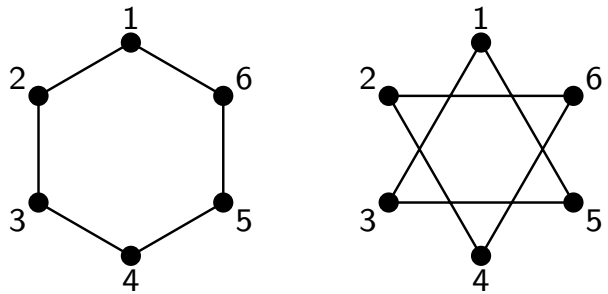
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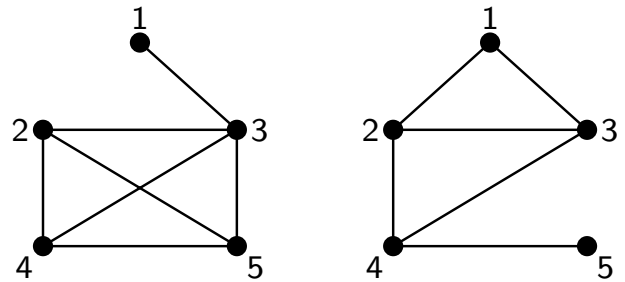
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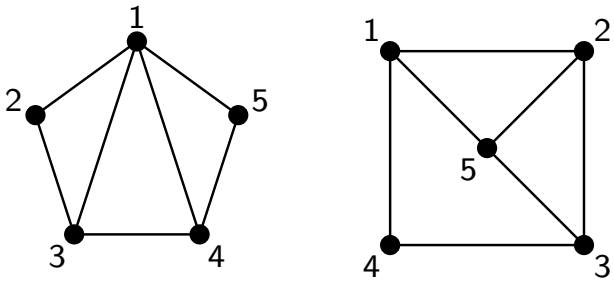
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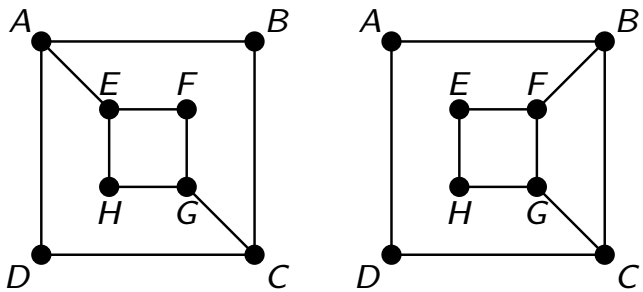
No: 1 vs. 2 components.



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No: maximum degrees 4 and 3.



No: 2 vs. 3 simple 4-cycles.

Graph Isomorphism

How to show two graphs are isomorphic:

- Find a bijection between the vertex sets that preserves adjacency.

How to show two graphs are not isomorphic:

- Find a structural property that differs between the two graphs (e.g., number of vertices, number of edges, degree sequence, number of components, presence of cycles, etc.).



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In general, it's not known what is the fastest way to determine if two graphs are isomorphic (take algos for more).



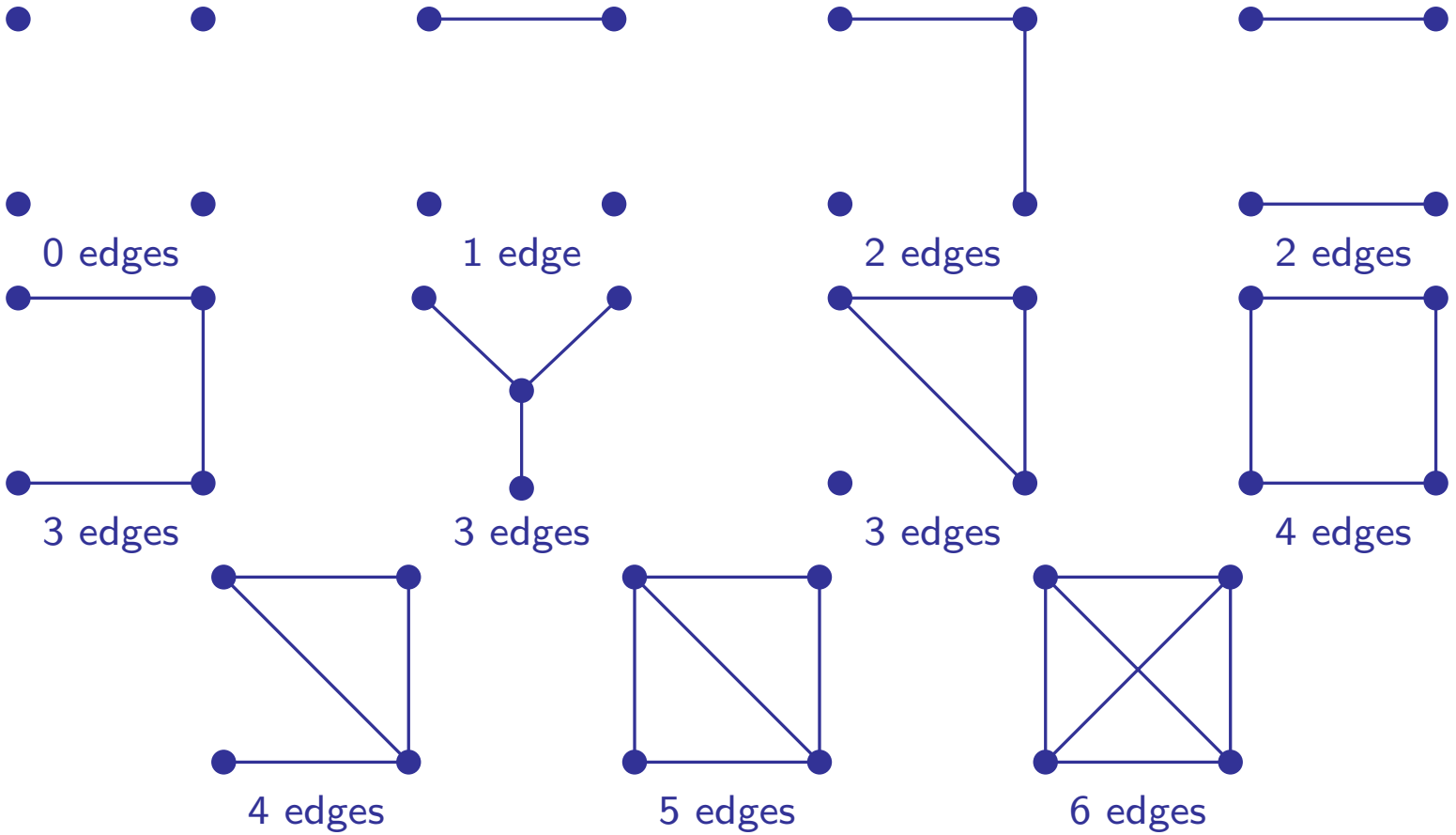
Graph Isomorphism: Exercises

Exercise: How many distinct graphs up to isomorphism are there on 4 vertices? Draw them all.

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Answer: There are 11 graphs up to isomorphism.



Graph Isomorphism: Exercises

Definition: The complement of $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where, for distinct $u, v \in V$,

$$\{u, v\} \in \overline{E} \quad \Longleftrightarrow \quad \{u, v\} \notin E.$$



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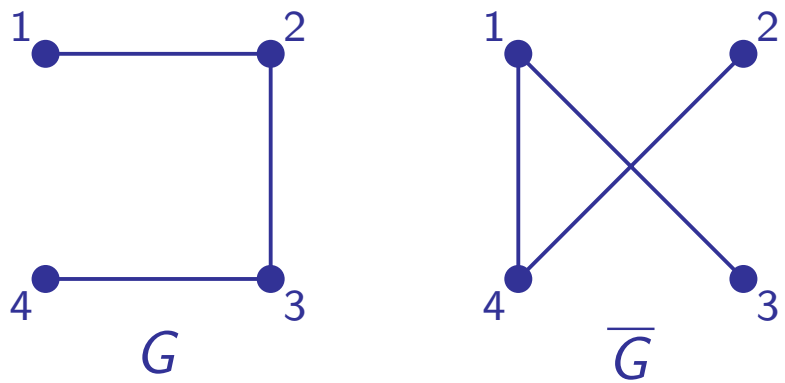
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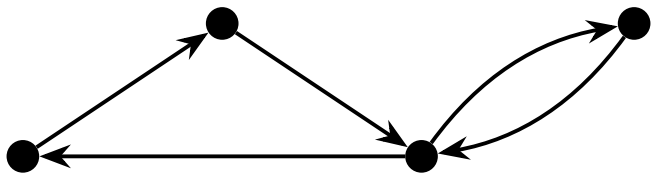
Answer: Yes: take a path on four vertices.



An isomorphism: $1 \mapsto 3, \quad 2 \mapsto 1, \quad 3 \mapsto 4, \quad 4 \mapsto 2.$

Graph Variants

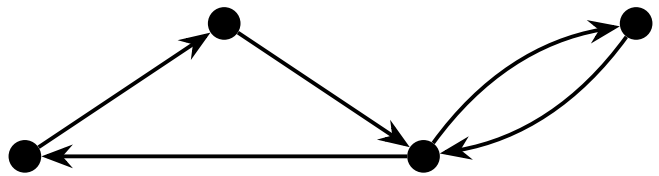
Directed graphs: Edges are ordered pairs (a, b), drawn as arrows. Vertices have an in-degree and an out-degree.



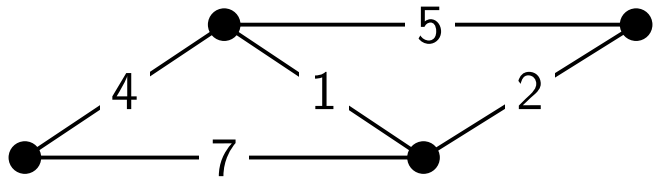


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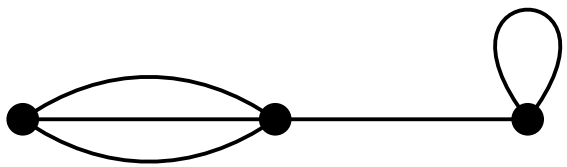
Weighted graphs: Edges carry numerical weights, such as distances or costs.





Graph Variants

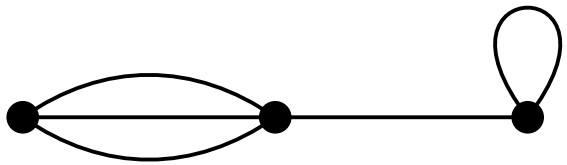
Multigraphs: Undirected graphs that allow multiple edges between the same pair of vertices and self-loops.





Graph Variants

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Hypergraphs: Edges are nonempty subsets of the vertex set, so an edge can contain more than two vertices.

