Start of Lecture 15

Last time: Graph Basics

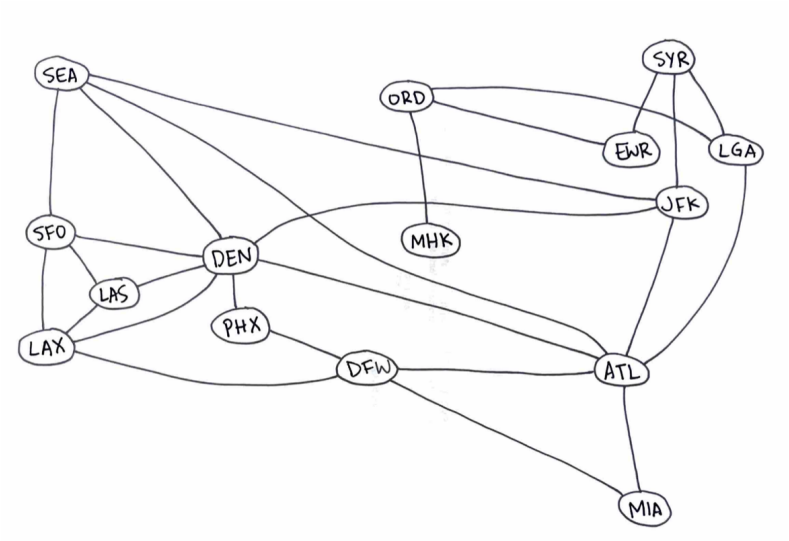
Today: Paths and Cycles



Recall: Graphs

Definition: A graph G is an ordered pair (V, E) where:

- V is a set of vertices.
- E is a set of edges, which are unordered pairs of distinct vertices from V.

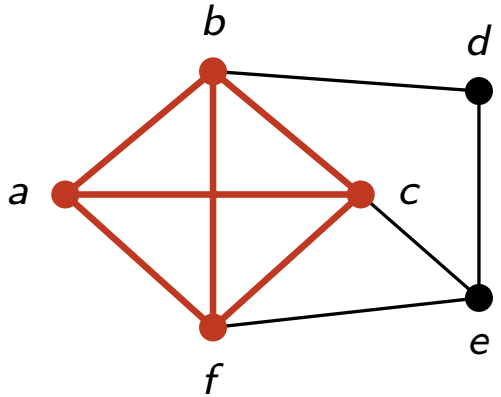


Recall: Cliques and Independent Sets



Definition: A clique is a subset of a graph's vertices that induces a complete graph.

Every pair of distinct vertices in the set is adjacent.

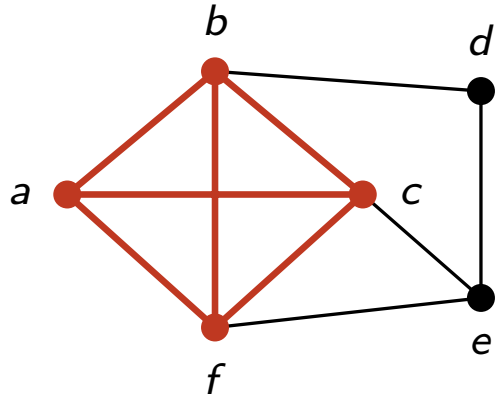


Recall: Cliques and Independent Sets



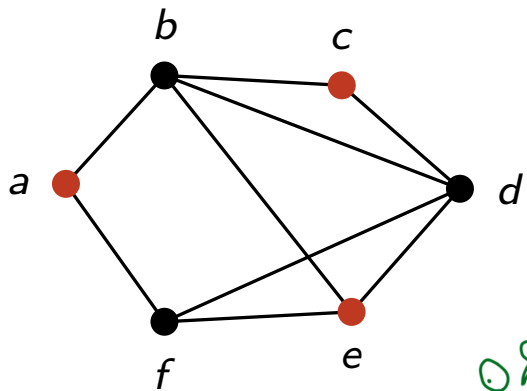
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Every pair of distinct vertices in the set is adjacent.



Definition: An independent set is a subset of a graph's vertices that induces an empty graph.

No two vertices in the set are adjacent.





Exercise



Question 1: Can you construct a graph on 5 vertices which has no clique of size 3 and no independent set of size 3?

Exercise



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Yes, the cycle graph is an example.

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Yes, the cycle graph is an example.

Question 2: Can you construct a graph on 6 vertices which has no clique of size 3 and no independent set of size 3?

No, this is impossible. Why?

Exercise: Proof

Claim: In any graph on 6 vertices, there must be either a clique of size 3 or an independent set of size 3.



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Proof: Let G be a graph on 6 vertices. Pick any vertex v. There are 5 other vertices.



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By the pigeonhole principle, at least 3 of them are connected to v or at least 3 of them are not connected to v.

Case 1: At least 3 of them are connected to v. If any two of them are connected to each other, we have a clique with v and those two vertices. If none of them are connected to each other, we have an independent set of size 3.



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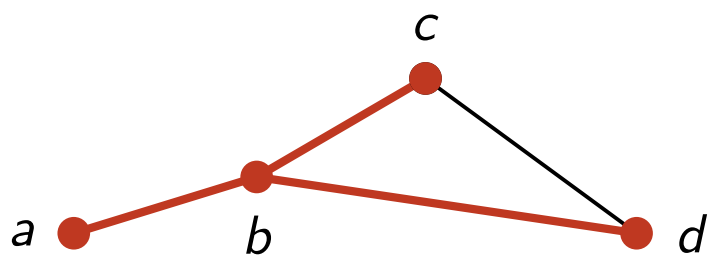


Paths

Definition: In a graph $G = (V, E)$, a path from u to v is a sequence of vertices

$$v_0 = u,\ v_1,\ v_2,\ \ldots,\ v_n = v$$

with $\{v_i, v_{i+1}\} \in E$ for every $0 \leq i < n$.



Path: a, b, c, b, d (repeats b).



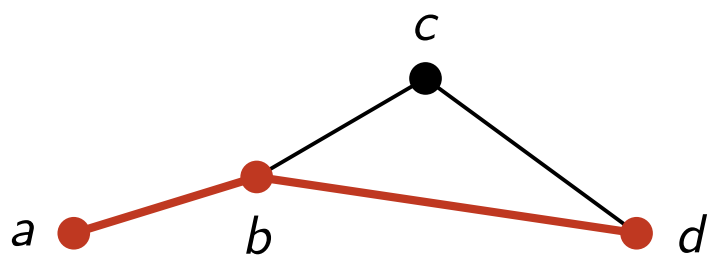
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Definition: A simple path is a path with no repeated vertices.



Simple path: a, b, d (no repeated vertices).



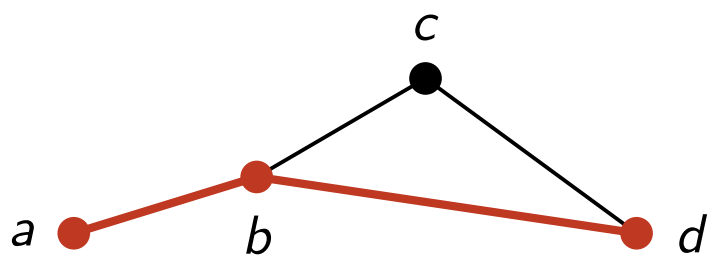
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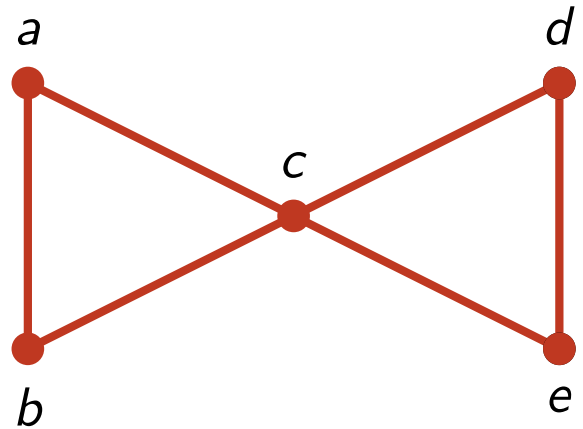
Simple path: a, b, d (no repeated vertices).



The length of a path is the number of edges in it.

Cycles

Definition: A cycle is a path from a vertex u back to itself, of length at least 1.



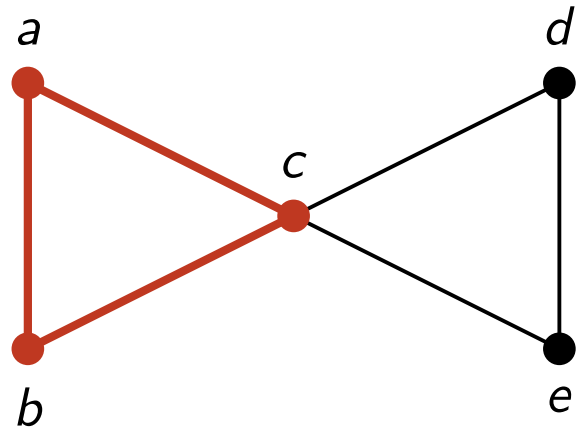


Cycle: c, a, b, c, d, e, c.

Cycles

Definition: A cycle is a path from a vertex u back to itself, of length at least 1.

Definition: A simple cycle is a cycle of length $k \geq 3$ in which $v_0, v_1, \ldots, v_{k-1}$ are all distinct.



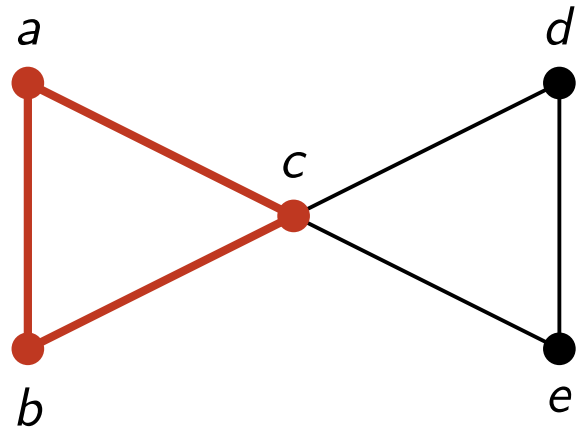


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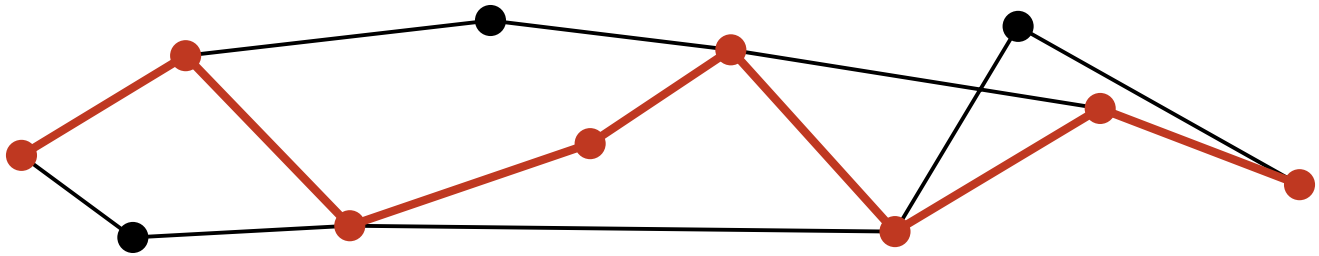


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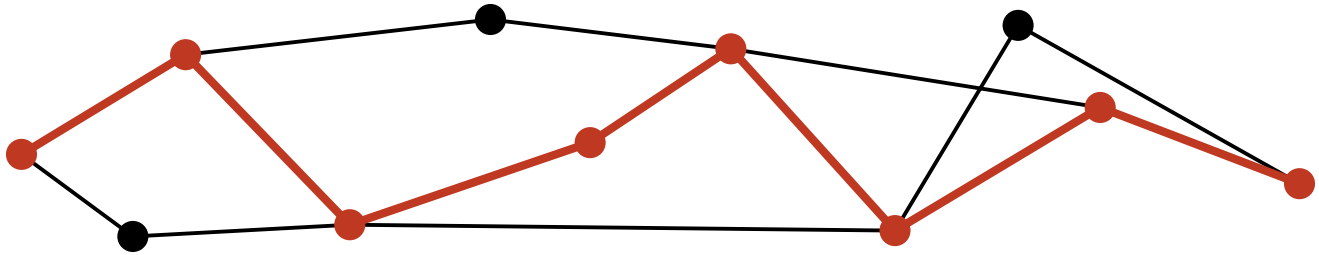
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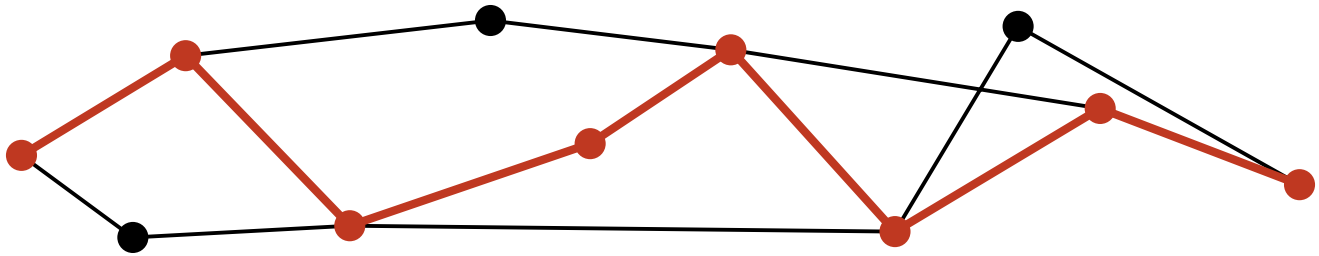


Proposition: Two vertices are connected if and only if there is a simple path between them.



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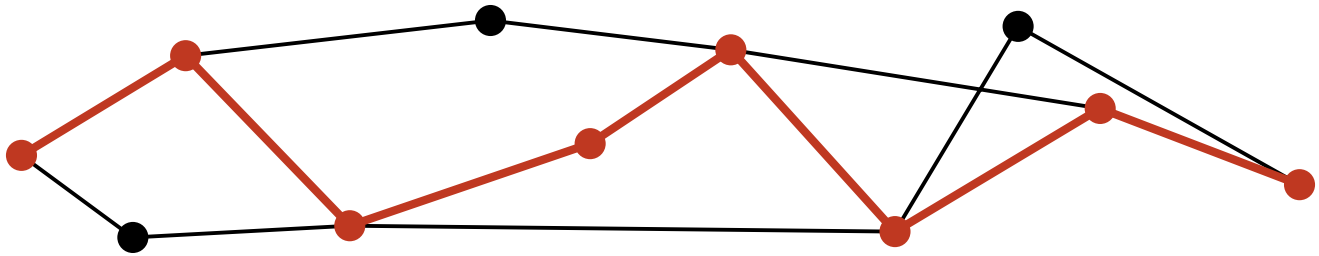
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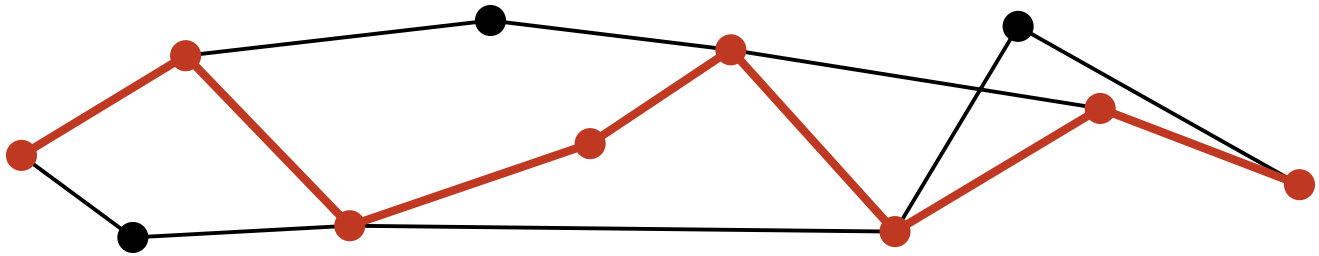
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($\Rightarrow$) Suppose u and v are connected. Choose a shortest path $v_0 = u, \ldots, v_n = v$.

If there was a repeat $v_i = v_j$ with $i < j$, deleting the steps between them gives

$$v_0, \ldots, v_i, v_{j+1}, \ldots, v_n.$$

This is a shorter path from u to v, a contradiction. Thus no vertex repeats, and the path is simple.



Connected Graphs and Components

Definition: A graph $G = (V, E)$ is connected if there is a path from u to v for every pair of distinct vertices $u, v \in V$.



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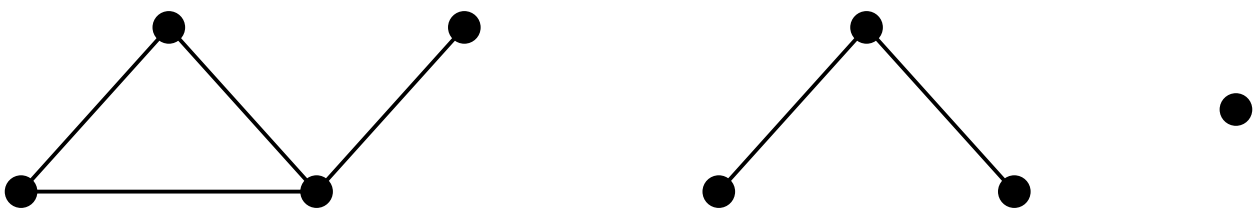


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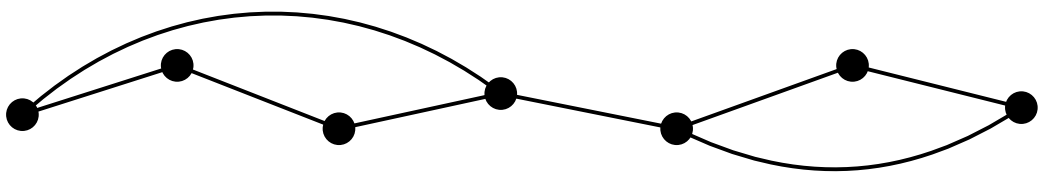
Example: How many connected components does this graph have?





Proving things with Longest Simple Paths

Proposition: If every vertex of G has degree at least 2, then G contains a simple cycle.

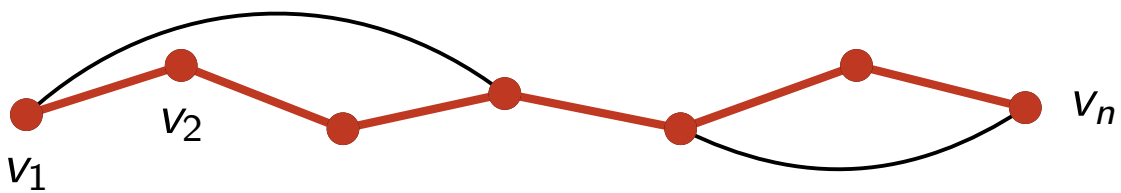




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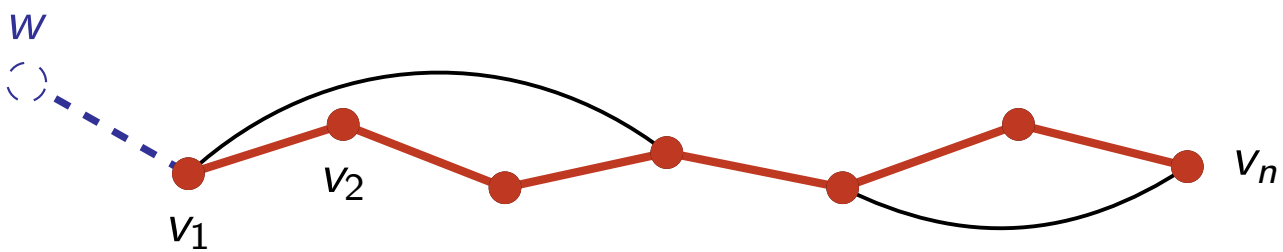


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Every neighbor of v_1 lies on this path: otherwise a neighbor w off the path would give a longer simple path $w, v_1, \ldots, v_n$.





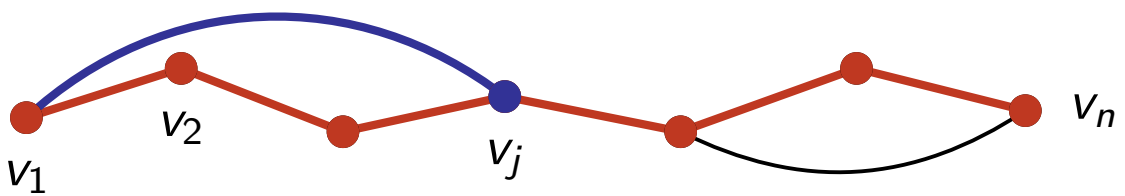
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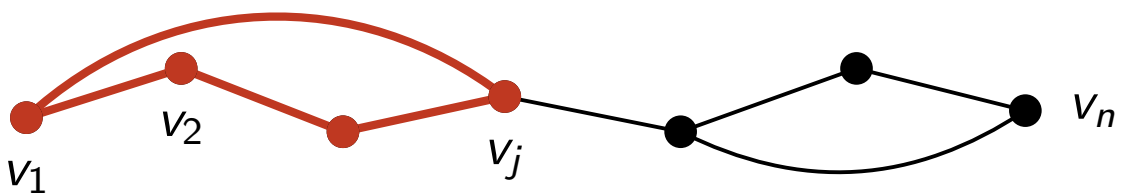
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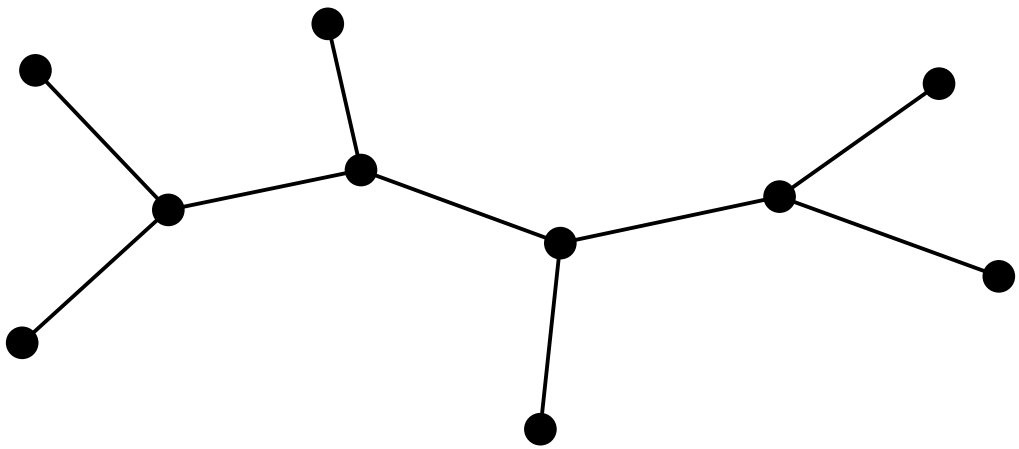
Thus $v_1, v_2, \ldots, v_j, v_1$ is a simple cycle.





Trees

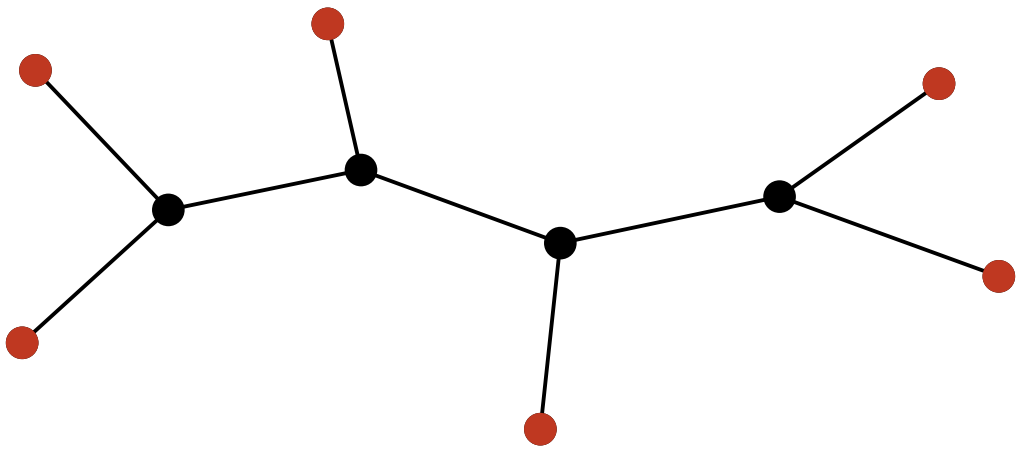
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Vertices of degree 1 are called leaves.



Trees have Leaves

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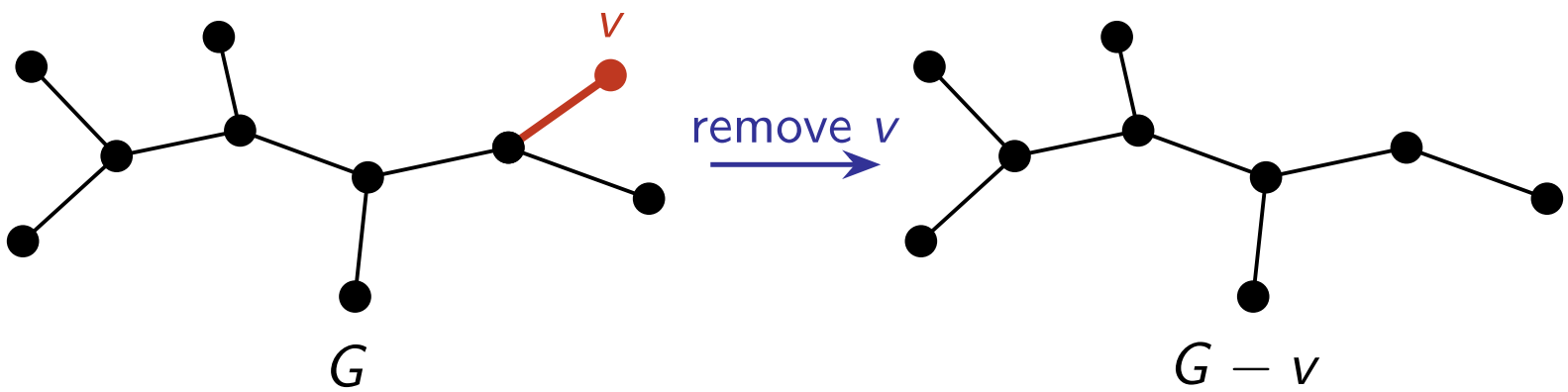
Proof: If there were no leaves, every vertex would have degree at least 2.

By the previous proposition, this would imply the existence of a simple cycle, contradicting the definition of a tree.



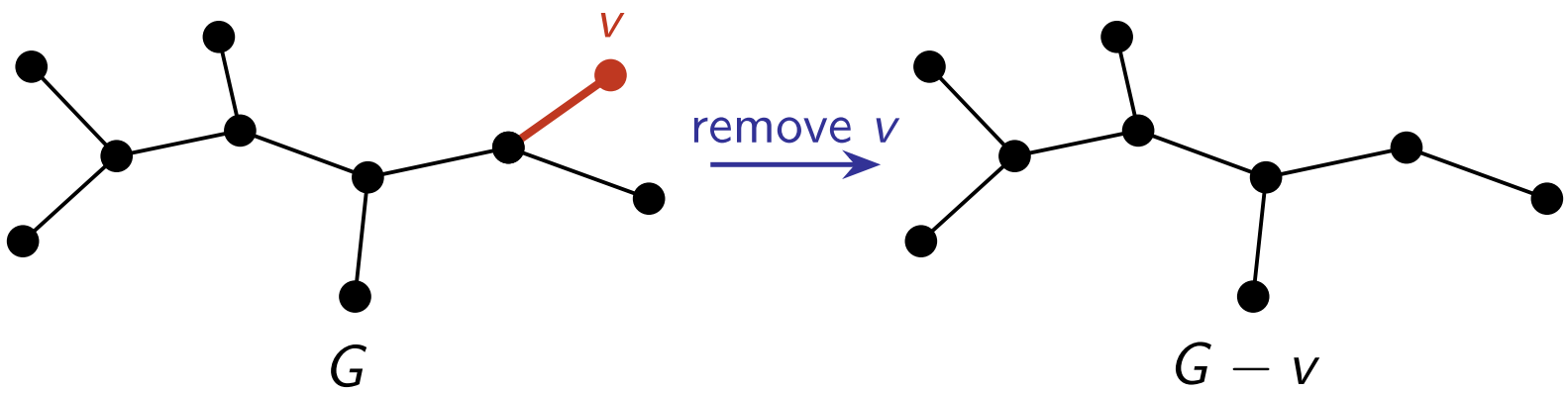
Trees

Fact: If we remove a leaf and all its edges from a tree, the remaining graph is still a tree.



Trees

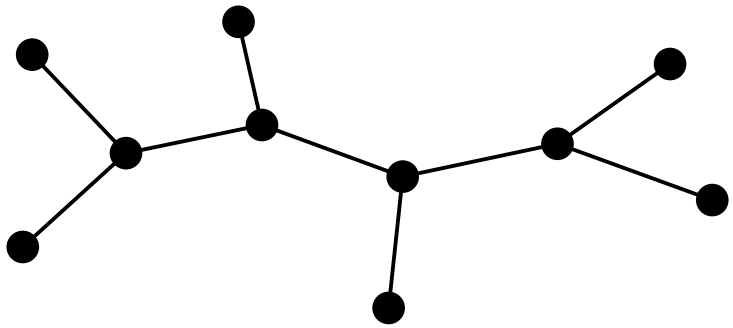
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Note: We'll use $G - v$ to denote the graph obtained from G by removing vertex v and all its edges.

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Proposition: Every tree is bipartite.





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Proof: By induction on the number of vertices n.

Base case: For $n = 1$, take $A = V$ and $B = \varnothing$.





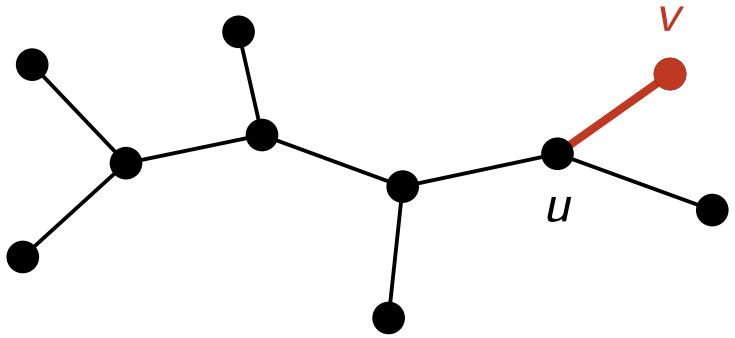
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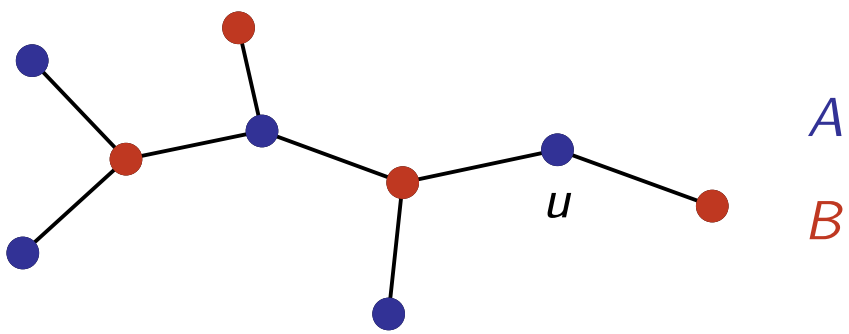
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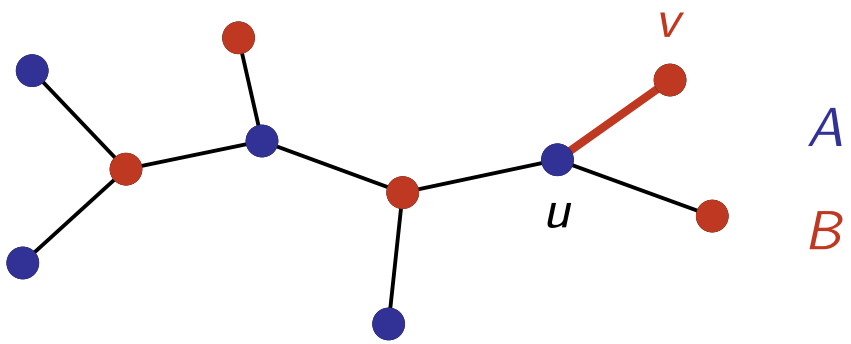
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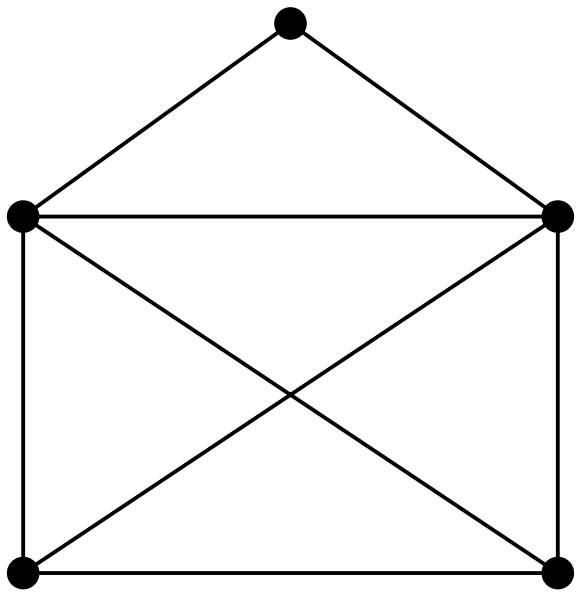
Say $u \in A$. Put v in B. The only new edge $\{u, v\}$ joins the two parts, so T is bipartite.





Eulerian Paths: Example

Example: Can you draw this graph without lifting your pencil and without drawing any edge twice?

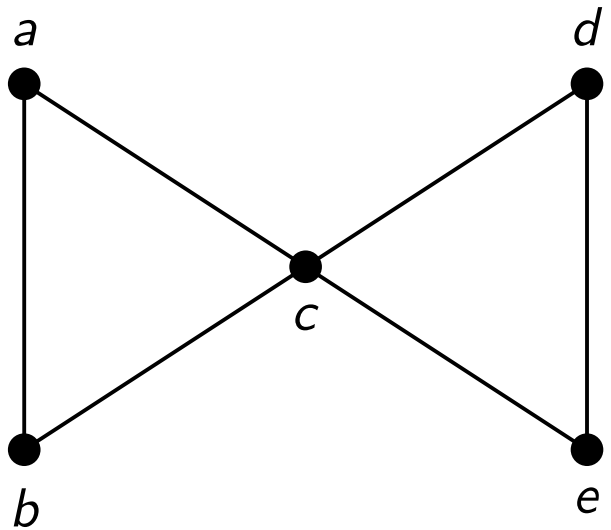




This is how far we got in class.

Eulerian Paths and Cycles

Definition: A path or cycle in a graph is Eulerian if it uses every edge of the graph exactly once.

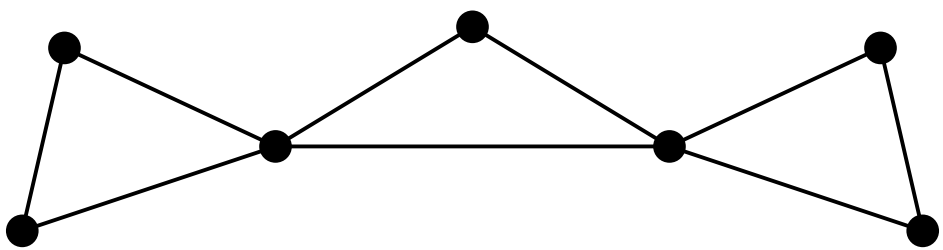


Eulerian cycle: c, a, b, c, d, e, c.

Eulerian Cycles



Theorem: A connected graph with at least one edge has an Eulerian cycle if and only if every vertex has even degree.

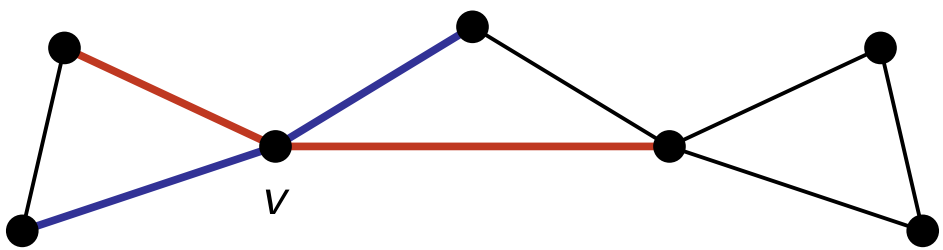


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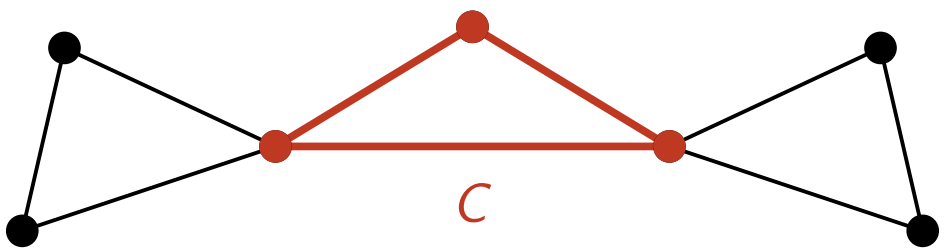
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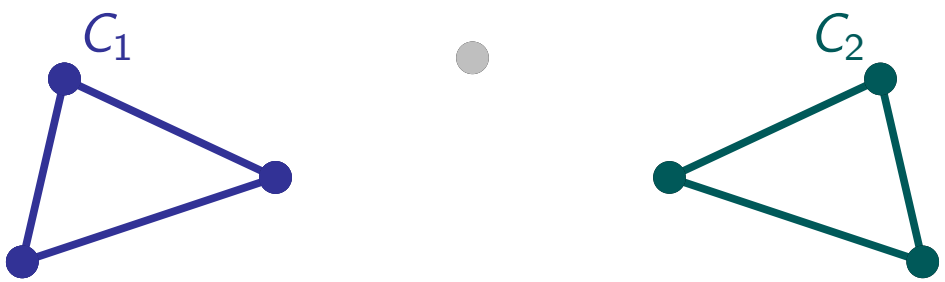


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Delete the edges of C. Degrees stay even. By induction, each remaining component with edges has an Eulerian cycle.



Eulerian Cycles



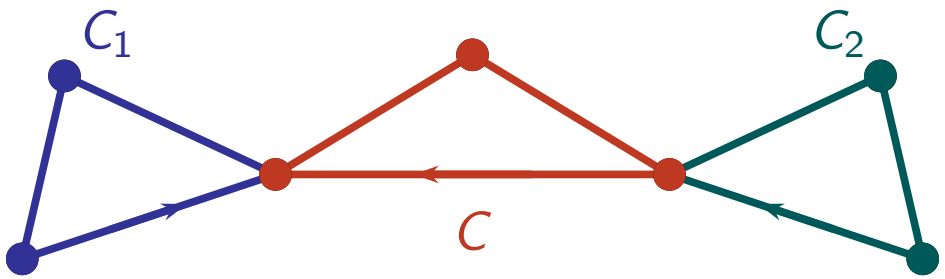
Theorem: A connected graph with at least one edge has an Eulerian cycle if and only if every vertex has even degree.

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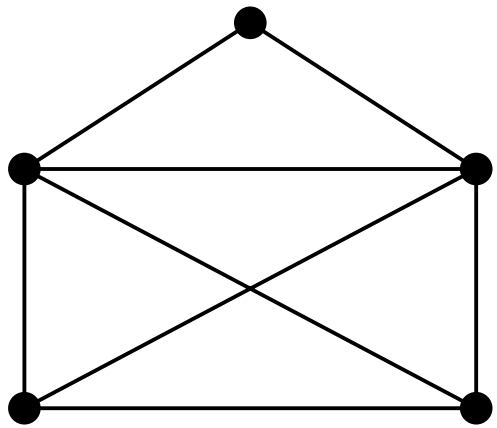
Delete the edges of C. Degrees stay even. By induction, each remaining component with edges has an Eulerian cycle.

Follow C, detouring around each component's Eulerian cycle. Every component meets C by connectedness, so every edge is used exactly once.



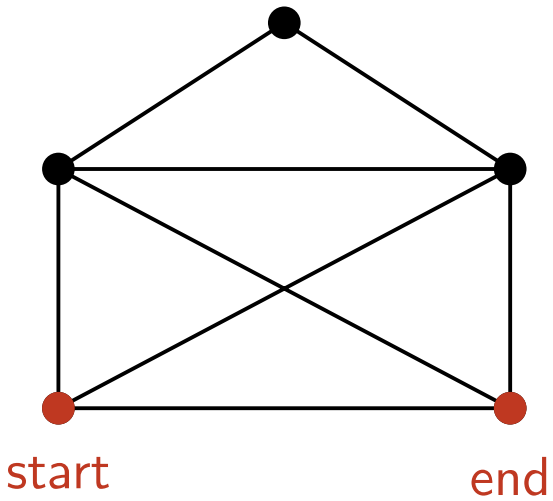
Eulerian Paths

Theorem: A connected graph has an Eulerian path if and only if it has exactly 0 or 2 vertices of odd degree.



Eulerian Paths

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Proof Idea: Same idea as Eulerian cycles proof but 2 odd degree vertices can be start and end of path.

Eulerian Cycles: Dominoes

Example: A domino set has $\binom{7}{2}$ dominoes, one for each pair of distinct numbers from 0 to 6.

Can you lay all of them out in a closed ring so that touching ends match?



Eulerian Cycles: Dominoes

Claim: The dominoes can be arranged in a closed ring so that all touching ends match.



Eulerian Cycles: Dominoes

Claim: The dominoes can be arranged in a closed ring so that all touching ends match.

Proof Idea: Let each number from 0 to 6 be a vertex, and each domino be an edge joining its two numbers.

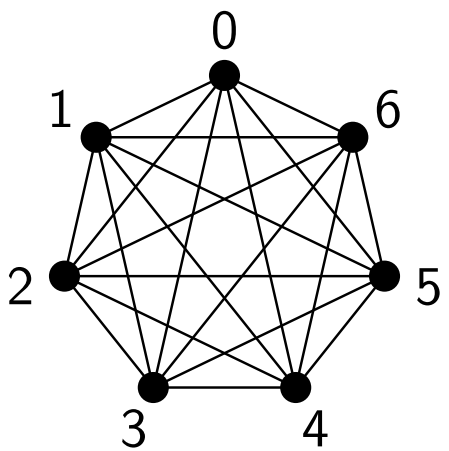


Eulerian Cycles: Dominoes

Claim: The dominoes can be arranged in a closed ring so that all touching ends match.

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This gives K_7, which is connected and has degree 6 at every vertex. Since every degree is even, it has an Eulerian cycle.





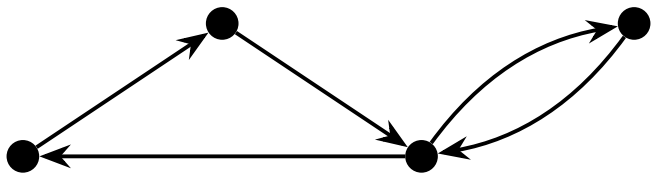


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Graph Variants

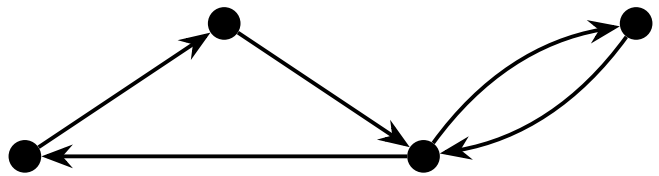
Directed graphs: Edges are ordered pairs (a, b), drawn as arrows. Vertices have an in-degree and an out-degree.





Graph Variants

Directed graphs: Edges are ordered pairs (a, b), drawn as arrows. Vertices have an in-degree and an out-degree.



Weighted graphs: Edges carry numerical weights, such as distances or costs.

