# Start of Lecture 14

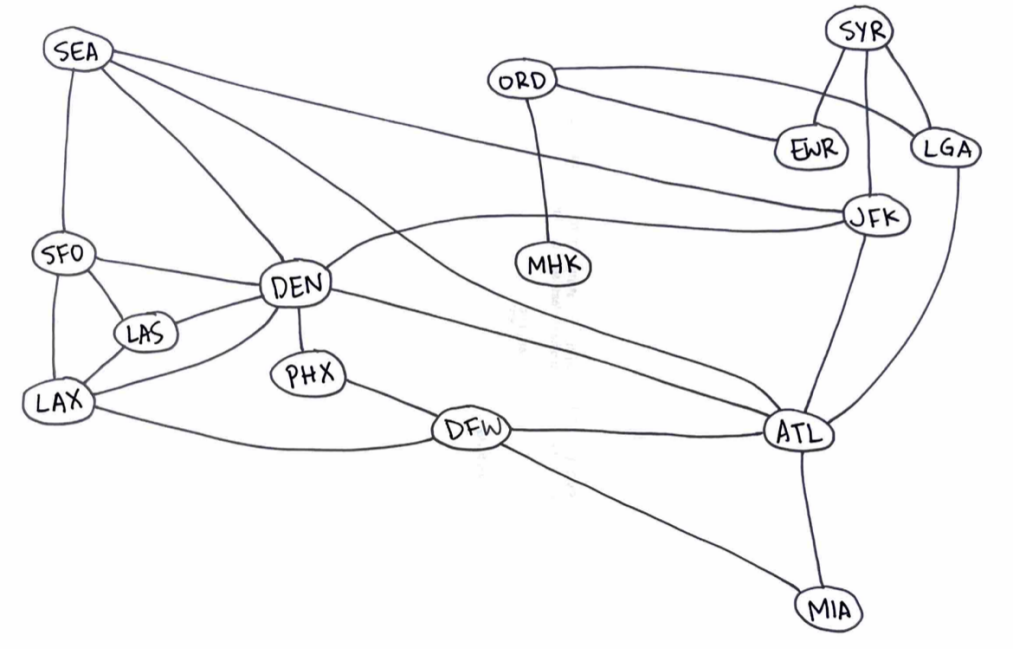
Last time: Pigeonhole Principle, Proof by Counting in Two Ways

Today: Graph Basics



# Graphs

**Graphs** are made up of vertices (nodes) and edges (connections between nodes).




Good for representing relationships, networks, connections, etc.

# Graphs: Definitions

**Definition:** A graph $G$ is an ordered pair $(V, E)$ where:



- $V$ is a set of vertices.
- $E$ is a set of edges, which are unordered pairs of distinct vertices from $V$

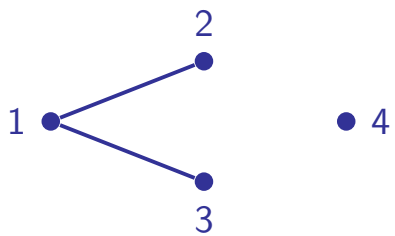
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**Example**: $V = [4]$ and $E = \{\{1, 2\}, \{1, 3\}\}$



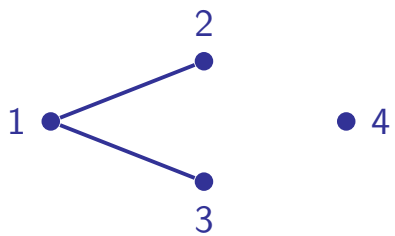
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**Example:**

$$V = \{\text{SEA}, \text{SFO}, \text{LAX}, \text{LAS}, \text{DEN}, \text{PHX}, \text{DFW}, \text{ATL}, \text{MIA}, \text{ORD}, \text{MHK}, \text{EWR}, \text{SYR}, \text{JFK}, \text{LGA}\},$$

$E = \{\{a, b\} : \text{there is a direct flight between } a \text{ and } b\}$.

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**Definition:** In a graph $G = (V, E)$, two vertices $u$ and $v$ are adjacent (aka. neighbors) if there is an edge connecting them.

Equivalently, $\{u, v\} \in E$.



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**Example:** SEA and JFK are adjacent in the airport graph.

# How to Define a Graph?

**How to define a graph?**

- Use a diagram.

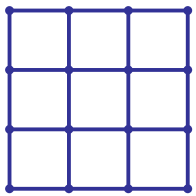


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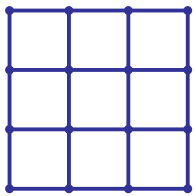


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- Use a formula and set-builder notation.

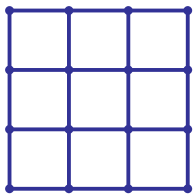


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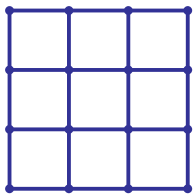


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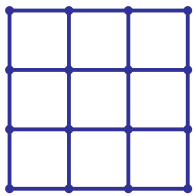


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  **Example:**

$$V = \{(1,1), (2,1), (3,1), (4,1), (1,2), (2,2), (3,2), (4,2), (1,3), (2,3), (3,3), (4,3), (1,4), (2,4), (3,4), (4,4)\},$$

$$E = \{\{(1,1),(2,1)\}, \{(2,1),(3,1)\}, \{(3,1),(4,1)\}, \{(1,2),(2,2)\}, \{(2,2),(3,2)\}, \{(3,2),(4,2)\}, \{(1,3),(2,3)\}, \{(2,3),(3,3)\}, \{(3,3),(4,3)\}, \{(1,4),(2,4)\}, \{(2,4),(3,4)\}, \{(3,4),(4,4)\}, \{(1,1),(1,2)\}, \{(1,2),(1,3)\}, \{(1,3),(1,4)\}, \{(2,1),(2,2)\}, \{(2,2),(2,3)\}, \{(2,3),(2,4)\}, \{(3,1),(3,2)\}, \{(3,2),(3,3)\}, \{(3,3),(3,4)\}, \{(4,1),(4,2)\}, \{(4,2),(4,3)\}, \{(4,3),(4,4)\}\}.$$



# Graphs: Definitions

**Definition:** The degree of a vertex $v$, denoted $\deg(v)$, is the number of neighbors of $v$.



Equivalently, $\deg(v) = |\{u \in V : \{u, v\} \in E\}|$.

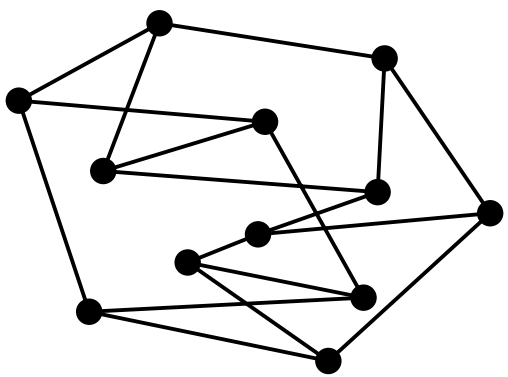
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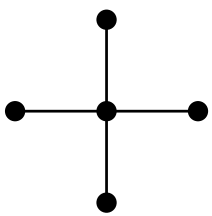
**Definition:** A regular graph is a graph where all vertices have the same degree.



# Graphs: Some Common Types

**Some common types of graphs:**

- the **star graph**: one central vertex joined to all the others;

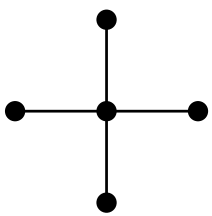


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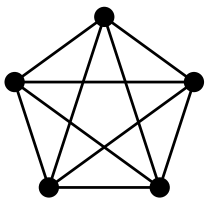
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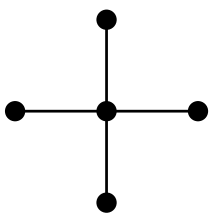


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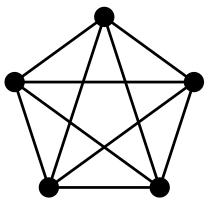
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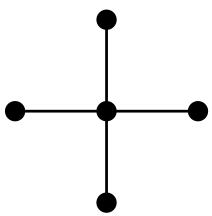


path

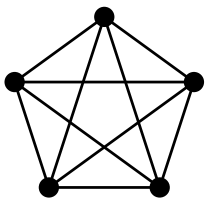
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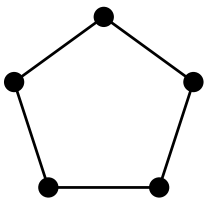
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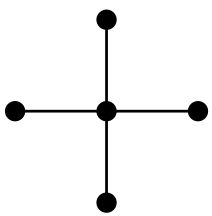


$C_5$

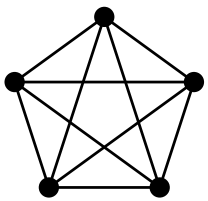
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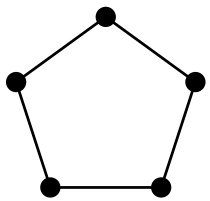
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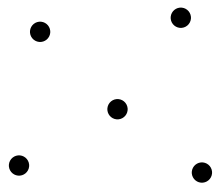
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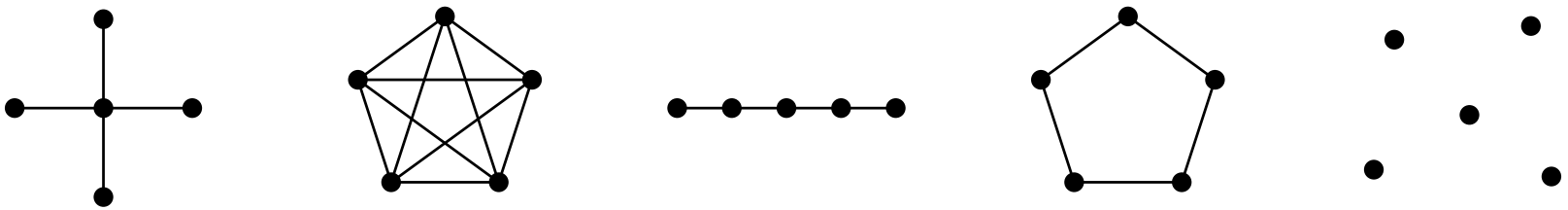


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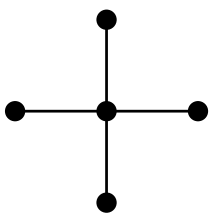



**Exercise:** For each of the examples above, how many edges do they have if there are $n$ vertices?

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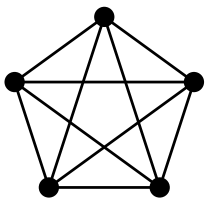
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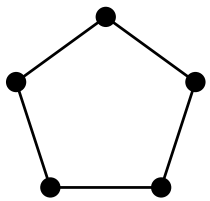
$n-1$



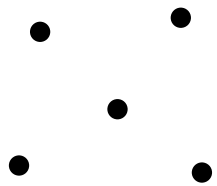
$K_5$



path



$C_5$



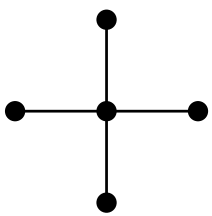
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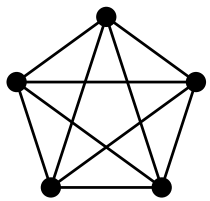
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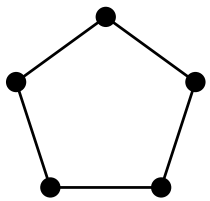


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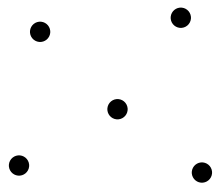
$\binom{n}{2}$



path



$C_5$



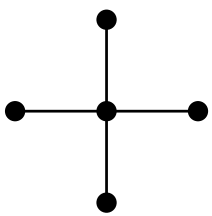
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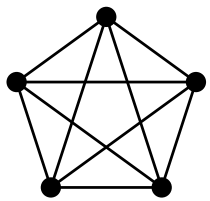
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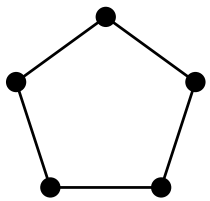
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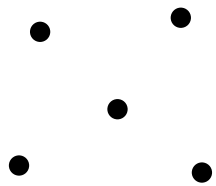


path

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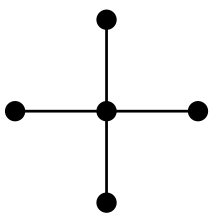
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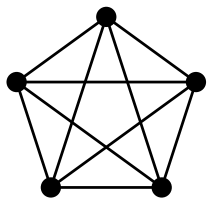
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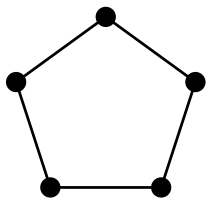
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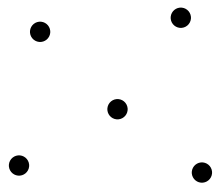
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$C_5$

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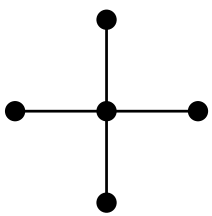
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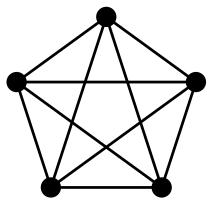
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star

$n-1$



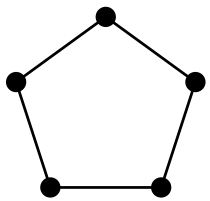
$K_5$

$\binom{n}{2}$



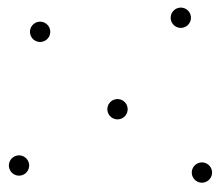
path

$n-1$



$C_5$

$n$



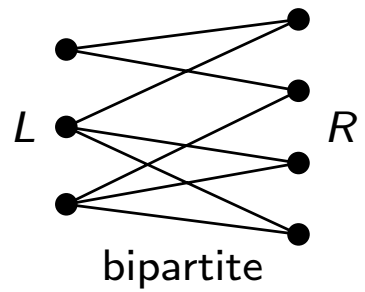
empty

0

**Exercise:** For each of the examples above, how many edges do they have if there are $n$ vertices?

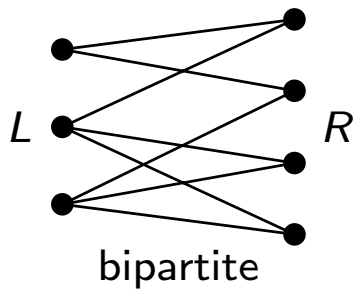
# Bipartite Graphs

**Definition:** A graph $G = (V, E)$ is bipartite if the vertex set $V$ can be partitioned into two disjoint sets $L$ and $R$ such that every edge in $E$ connects a vertex in $L$ to a vertex in $R$.

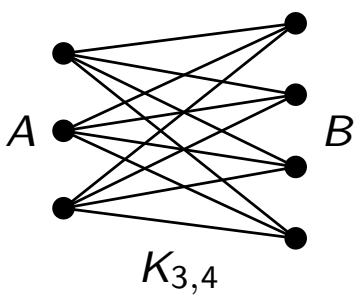


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**Definition:** The complete bipartite graph has every possible edge between the two sets of vertices with $|A| = m$ and $|B| = n$, denoted $K_{m,n}$.



# Handshake Lemma

**Theorem:** In any graph $G = (V, E)$

$$\sum_{v \in V} \deg(v) = 2|E|$$



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**LHS:** Each vertex $v$ contributes $\deg(v)$ incidences, so the total is $\sum_{v \in V} \deg(v)$.

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Remember shaking hands in lecture 1?

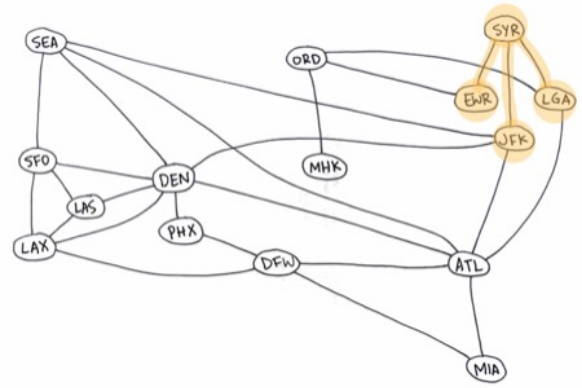
# Subgraphs

**Definition:** A subgraph $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

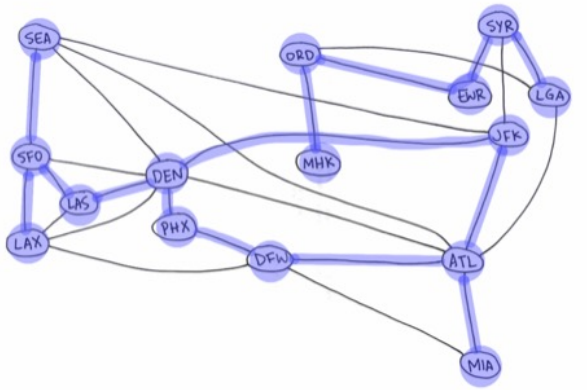
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**Examples:**




$G_1$: Just the New York airports and flights.

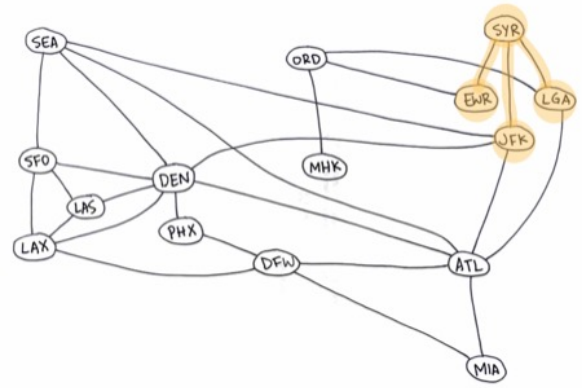



$G_2$: Airports and flights serviced by United.

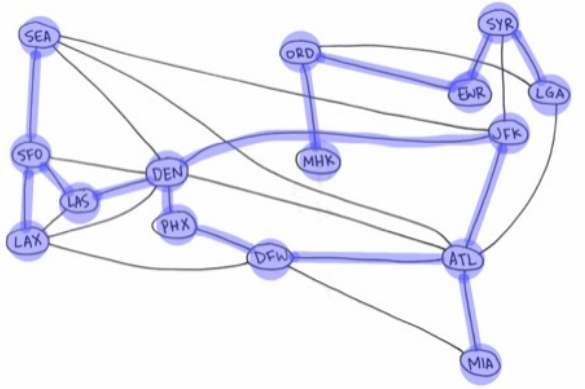
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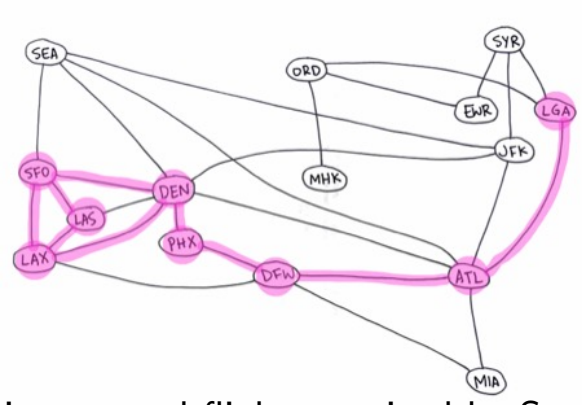
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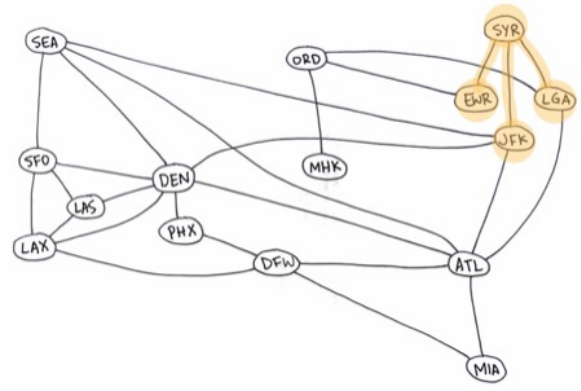



$G_3$: Airports and flights serviced by Southwest

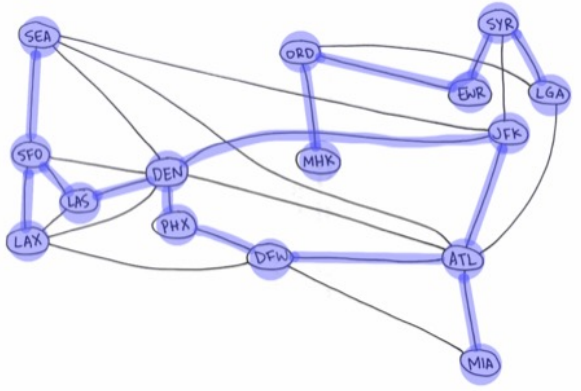
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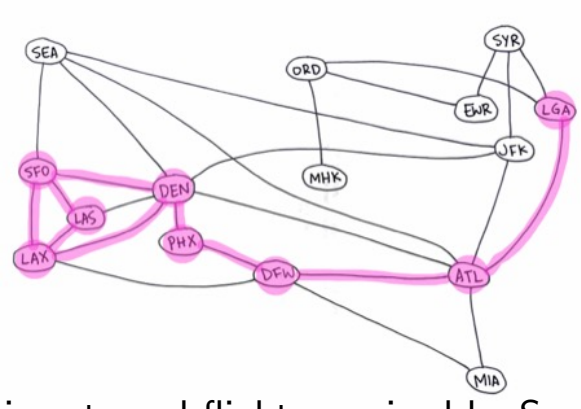
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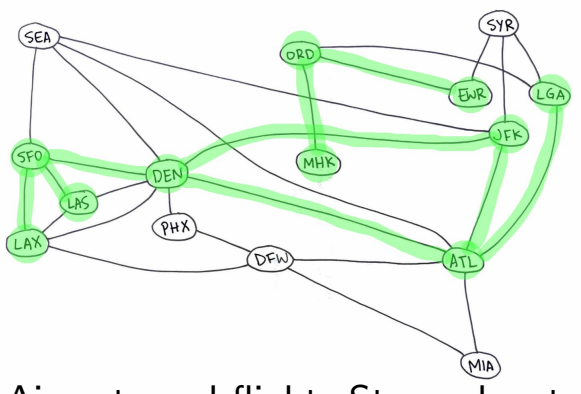

$G_1$: Just the New York airports and flights.




$G_2$: Airports and flights serviced by United.




$G_3$: Airports and flights serviced by Southwest




$G_4$: Airports and flights Styopa has taken.

# Subgraphs: Spanning

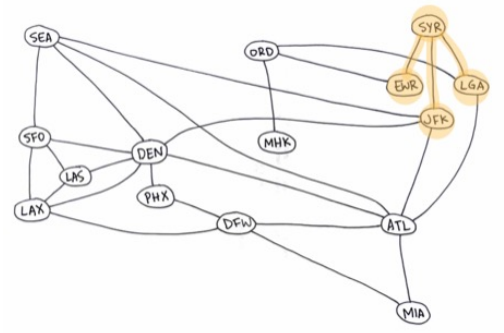
**Definition:** A subgraph is spanning if it contains all the vertices of the original graph, (i.e., $V' = V$.)



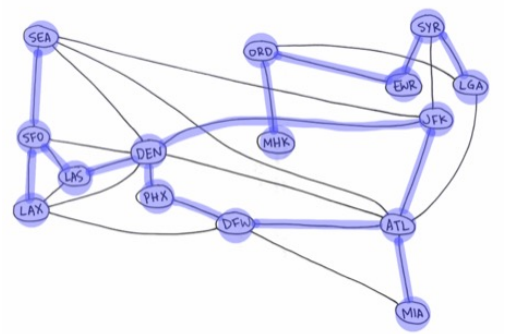
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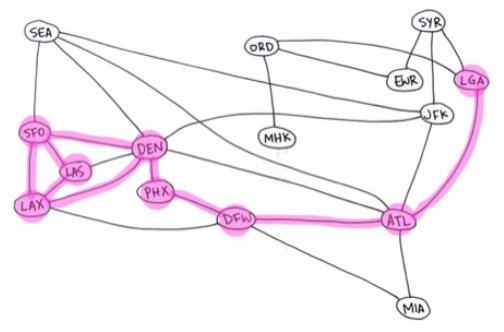
Which of the 4 examples of subgraphs are spanning subgraphs?



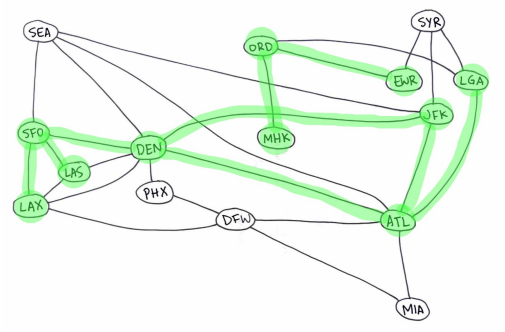

$G_1$: New York




$G_2$: United




$G_3$: Southwest




$G_4$: Styopa



# Subgraphs: Induced

**Definition:**

For a graph $G = (V, E)$ the induced subgraph on a set of vertices $V' \subseteq V$ is $(V', E')$ where $E' = \{\{u, v\} \in E : u, v \in V'\}$.

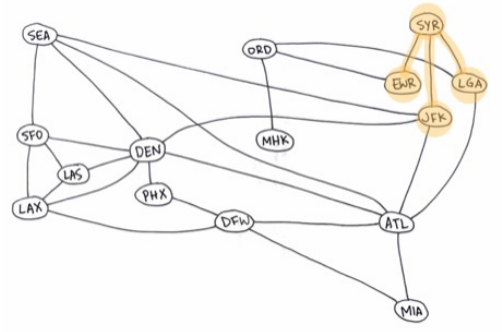


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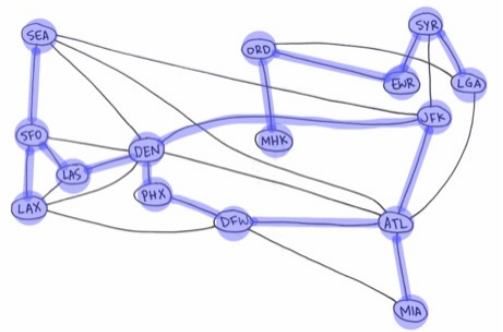
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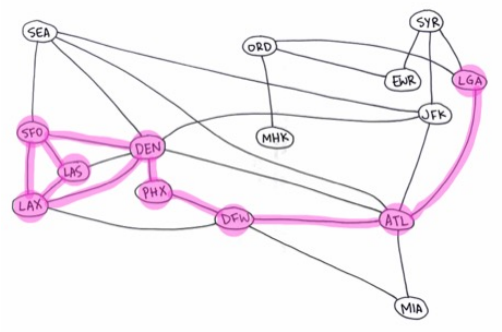
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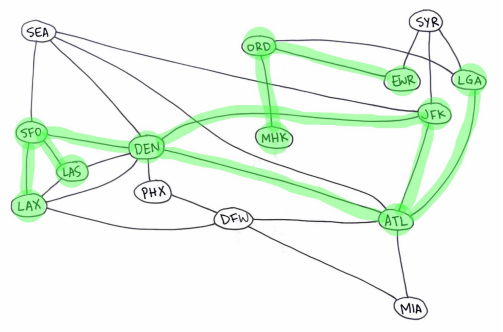

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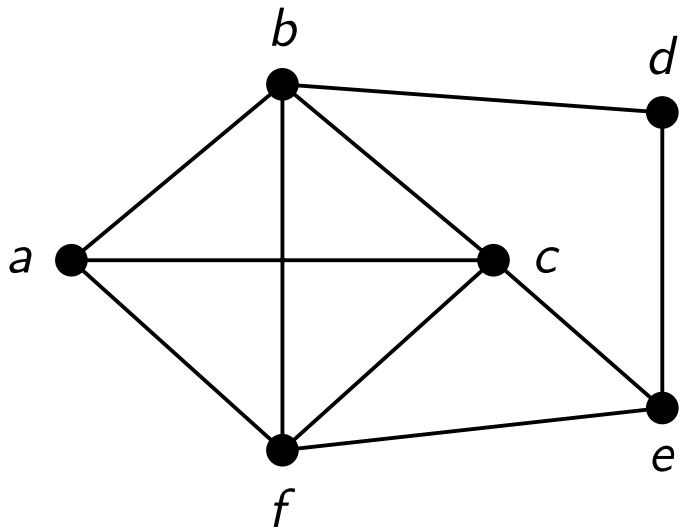



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# Subgraphs: Cliques

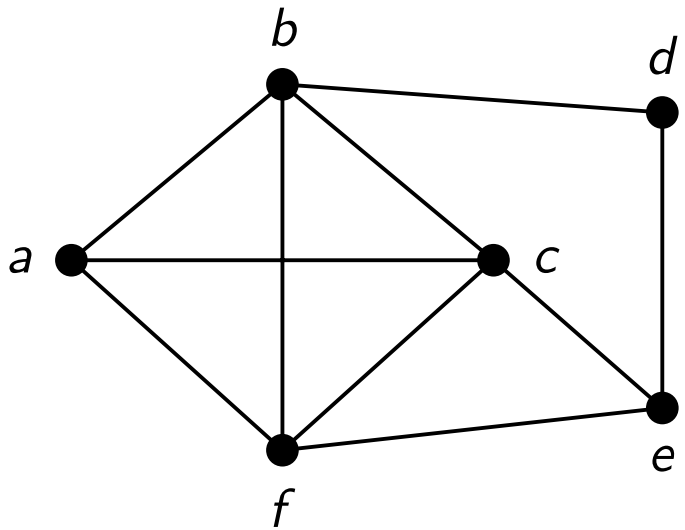
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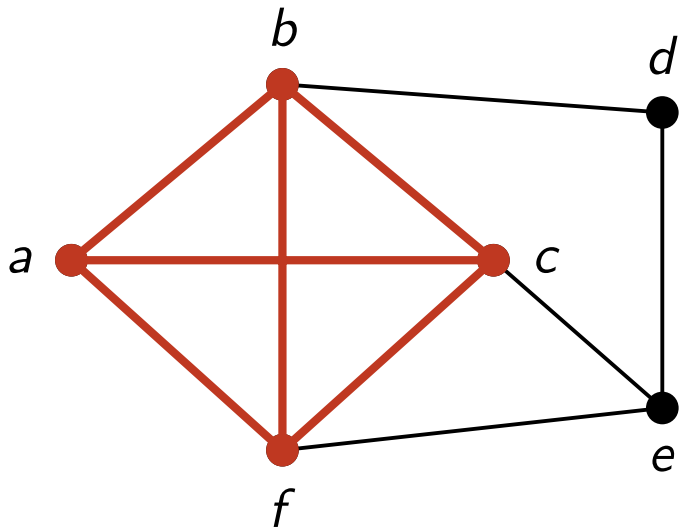



What is the largest clique in the graph above?



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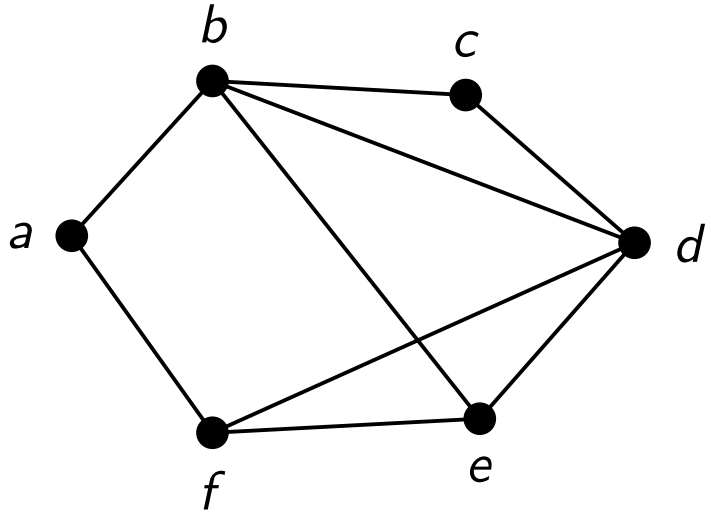

What is the largest clique in the graph above?

The highlighted vertices $\{a, b, c, f\}$ form a clique of size 4.



# Subgraphs: Independent Sets

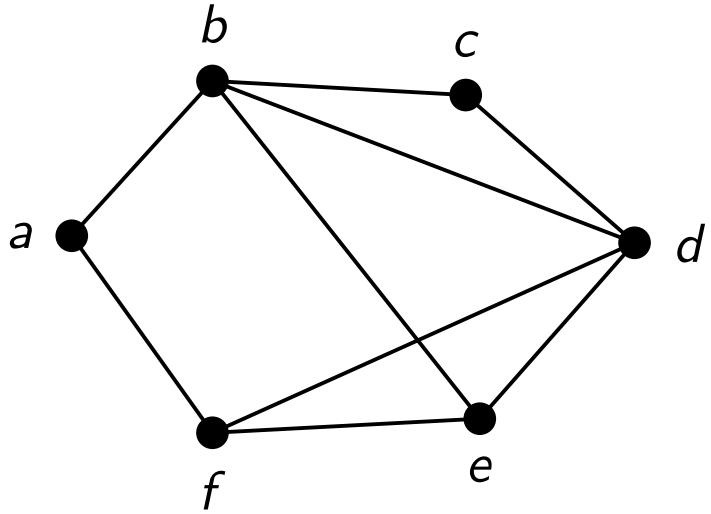
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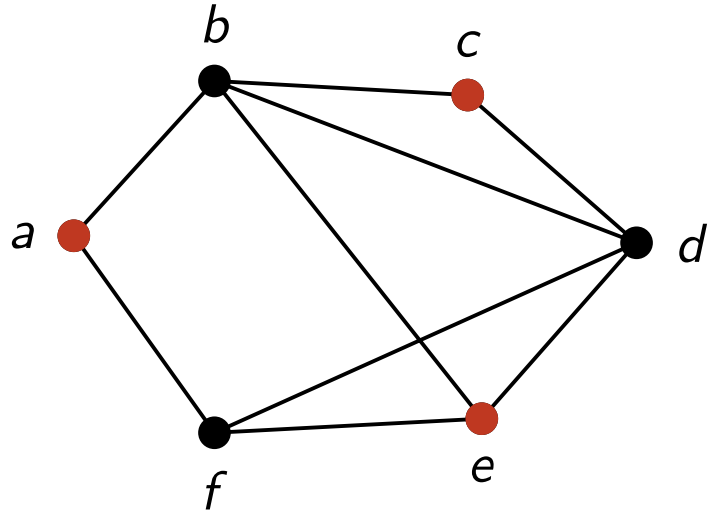



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# Subgraphs: Independent Sets

**Definition:** An independent set is a subset of a graph's vertices that induces an empty graph.




What is the largest independent set in the graph above?

The highlighted set $\{a, c, e\}$ is an independent set of size 3.



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In a graph with $n \geq 2$ vertices, it's impossible to have both a vertex of degree 0 and a vertex of degree $n-1$.

Therefore, there are at most $n-1$ possible degrees for $n$ vertices. By the Pigeonhole Principle, at least two vertices must share the same degree.

This is how far we got in this lecture.

# Exercise 2



**Question 1:** Can you construct a graph on 5 vertices which has no clique of size 3 and no independent set of size 3?

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Yes, the cycle graph is an example.

**Question 2:** Can you construct a graph on 6 vertices which has no clique of size 3 and no independent set of size 3?

No, this is impossible. Why?

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**Claim:** In any graph on 6 vertices, there must be either a clique of size 3 or an independent set of size 3.



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