Start of Lecture 13

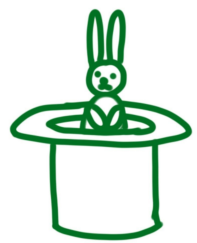
Last time: Counting, Balls and Walls

Today: Pigeonhole Principle, Proof by Counting in Two Ways



Pigeonhole Principle

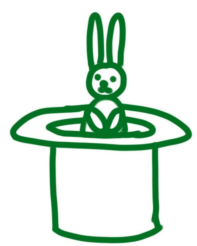
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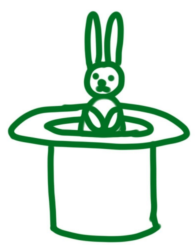


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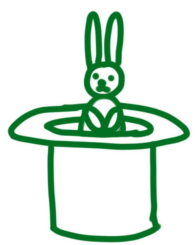
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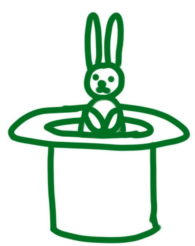
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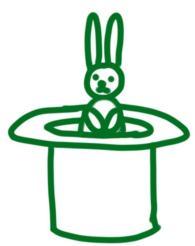
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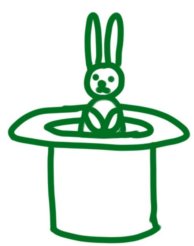
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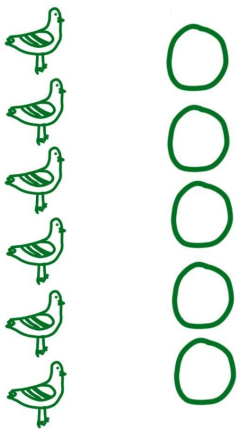
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So, at least two subsets of S must have the same sum.



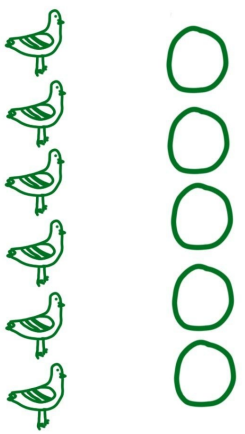
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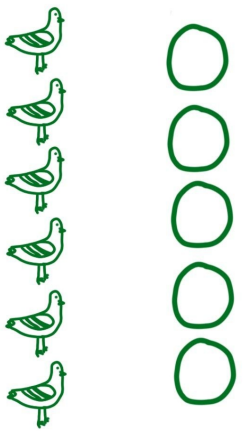
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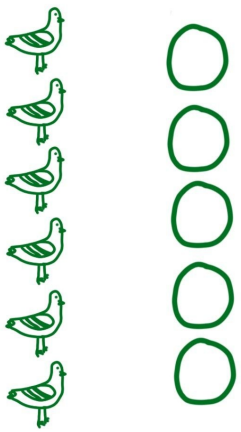
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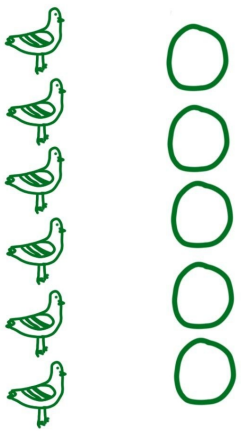
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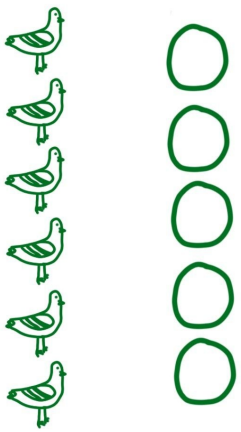
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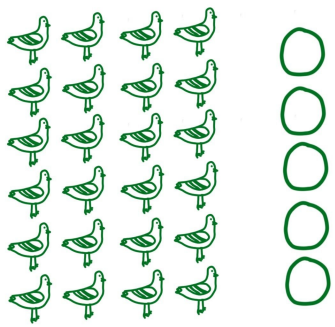
If more pigeons than holes are available, some hole must contain at least two pigeons.

Example: If 13 people are in a room, at least two of them have the same birth month.

Pigeonhole Principle: Generalized version

Generalized Pigeonhole Principle: Let A and B be finite sets. If $|A| > k|B|$, then for any $f : A \to B$, there exists some $b \in B$ such that

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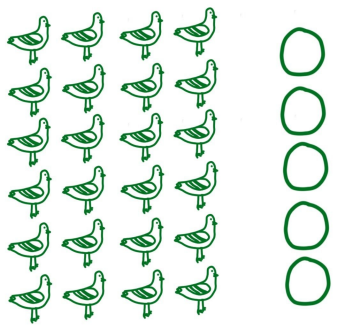


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Example: Everyone has at most 200k hairs on their head. There are 8.5 million people in NYC. So there's a group of at least 43 people in NYC with the exact same number of hairs on their head.



Proofs by Counting in 2 Ways

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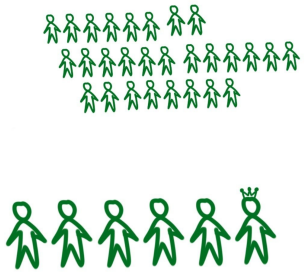
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Proofs by Counting in 2 Ways: Example 1

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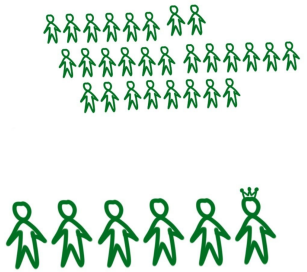
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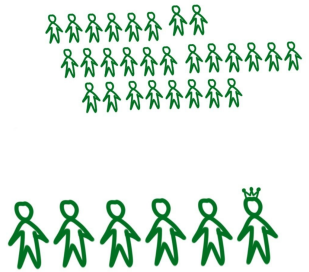


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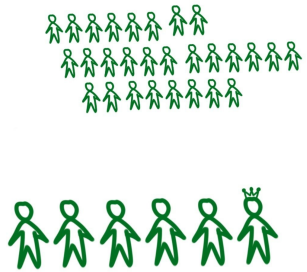
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RHS:

- Choose a chairperson from n people: n ways
- Choose the remaining $k-1$ committee members from the remaining people: $\binom{n-1}{k-1}$ ways

Proofs by Counting in 2 Ways: Example 2

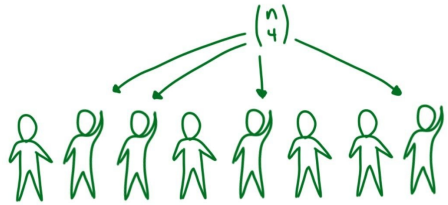
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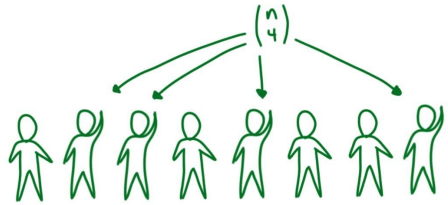


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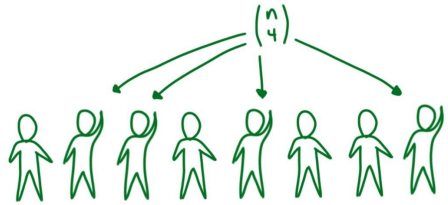
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RHS:

- Partition the subsets according to their size i
- There are $\binom{n}{i}$ subsets of size i
- By the addition principle, there are $\sum_{i=0}^{n} \binom{n}{i}$ subsets

Proofs by Counting in 2 Ways: Exercises

Counting in 2 Ways Exercises: Prove the following identities

- $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

- $$\binom{n}{a}\binom{n-a}{b} = \binom{n}{a+b}\binom{a+b}{a}.$$



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  Proof Idea: Case by if the nth element is in the subset.

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  Proof Idea: HW 6



Counting Exercises

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- How many 5-digit zip codes contain at least 3 different numbers?
- How many shortest paths from the bottom left corner to the top right corner of a $2n \times 2n$ grid that don't go through the center point?
- How many hands of 6 cards contain no kings or no queens or no diamonds?
- How many 10-digit phone numbers have their digits in non-decreasing order?

