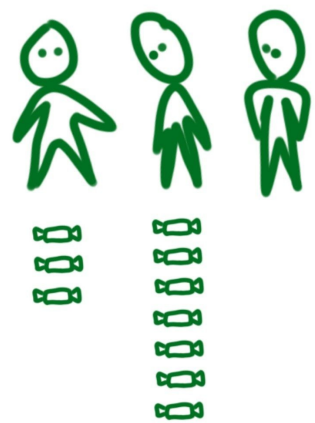
Start of Lecture 12

Last time: Counting

Today: Counting, Balls and Walls



Recap: Counting

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Counting questions ask "how many objects are in a set?"

e.g. You have 12 pants, 8 shirts, 5 dresses, and 3 shoes. How many different outfits can you make?



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e.g. You have 12 pants, 8 shirts, 5 dresses, and 3 shoes. How many different outfits can you make?

Techniques so far:

- Create a sequence of choices that uniquely identifies each object. Then use the multiplication principle.
- Partition the set into cases, count the number of objects in each case, and then use the addition principle.



Recap: Permutations, Binomial Coefficients

Recap:

- The number of permutations of n objects is $n!$. (via the multiplication principle).





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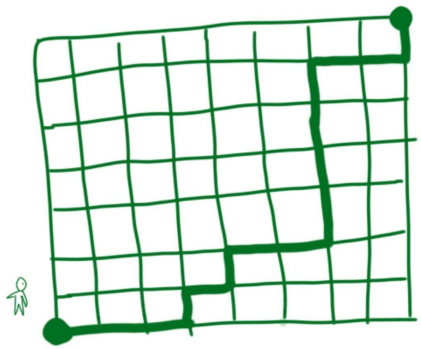
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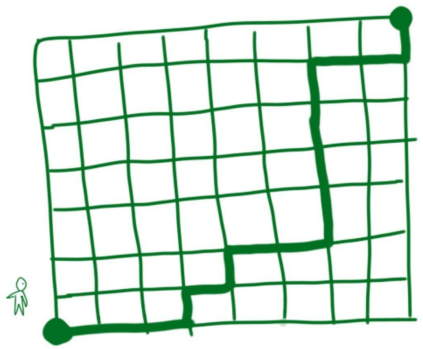
Recap: Example 2

Example 2: How many ways are there to go from the bottom left corner to the top right corner of an $m \times n$ grid, if you can only move up or right?



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Solution: $\binom{m+n}{m}$. Each path consists of m moves up and n moves right, for a total of $m+n$ moves. The number of ways to choose which m of the $m+n$ moves are up is $\binom{m+n}{m}$. This choice uniquely specifies the path and every path can be specified in this way.

Example 3

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Alt. Solution Idea: $\frac{11!}{4!4!2!1!}$. There are 11! ways to arrange the letters if they were all distinct, but we have overcounted by a factor of 4! for the I's, 4! for the S's, 2! for the P's, and 1! for the M.

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Counting Techniques

Counting techniques so far:

- Uniquely specify each element with a sequence of choices and use the multiplication principle.
- Partition the set into disjoint subsets and use the addition principle.
- Overcount and divide by the number of times each element is counted.
- Overcount and subtract the number of elements that are counted too many times.



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Question: Treat a phone number as a string of 10 digits. How many phone numbers contain exactly 3 zeros?

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We can uniquely specify each such phone number by choosing which 3 of the 10 positions are zeros, and then filling in the remaining 7 positions with any of the 9 remaining possible digits.



Example 6

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There are $\binom{10}{3}\binom{7}{3}8^4$ phone numbers with exactly 3 zeros and exactly 3 ones because each such phone number can be uniquely specified by choosing which 3 of the 10 positions are zeros, then which 3 of the remaining 7 positions are ones, and then filling in the remaining 4 positions with any of the 8 remaining possible digits.



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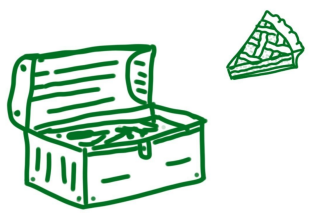
Therefore, the final answer is $\binom{10}{3}9^7 + \binom{10}{3}9^7 - \binom{10}{3}\binom{7}{3}8^4$.

Recall: Principle of Inclusion-Exclusion

Recall Principle of Inclusion-Exclusion: If A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Helpful because $|A \cup B|$ is often harder to count than $|A \cap B|$



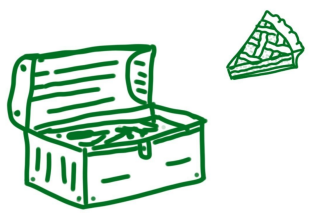
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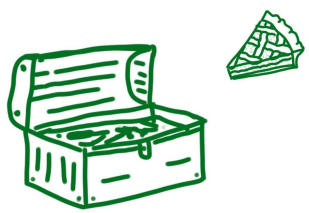
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Fact: If A, B, and C are finite sets, then

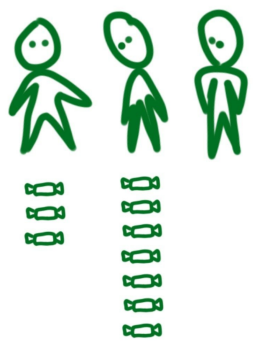
$$\begin{aligned} |A \cup B \cup C| = {} & |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C|. \end{aligned}$$



Balls and Walls

Question: I have m identical candies and n different children. How many ways are there to distribute the candies to the children?

e.g. $m = 10$, $n = 3$, I can give 3 candies to the first child, 7 candies to the second child, and 0 candies to the third child.



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There are $\binom{m+n-1}{n-1}$ ways to choose which $n-1$ of the $m+n-1$ positions are dividers, so there are that many ways to distribute the candies.

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This technique is known as "balls and walls", but also as "stars and bars" or "sticks and stones" or "balls and urns" or ...

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- Use the principle of inclusion-exclusion to count the size of intersections instead of unions.
- Use the balls and walls technique to count distributions of identical objects into distinct boxes.



Exercises

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- How many 5-digit zip codes contain at least 3 different numbers?
- How many shortest paths from the bottom left corner to the top right corner of a $2n \times 2n$ grid that don't go through the center point?
- How many hands of 6 cards contain no kings or no queens or no diamonds?
- How many 10-digit phone numbers have their digits in non-decreasing order?

