Start of Lecture 11

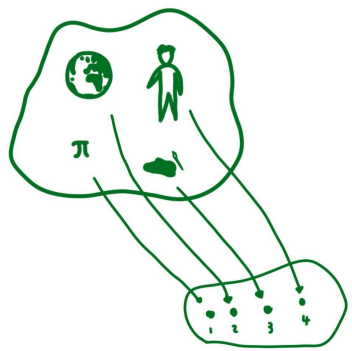
Last time: Uncountability, Cardinality

Today: Counting



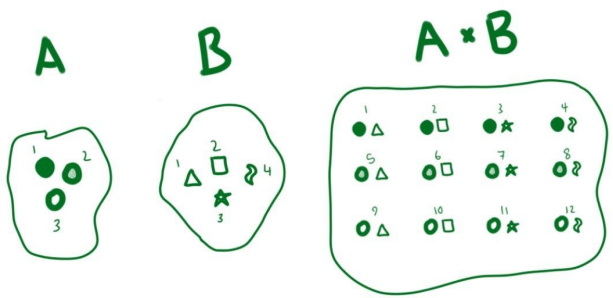
Recap: Size of a set

A set S has size n if there is a bijection between S and $[n]$.



Size of a Direct Product

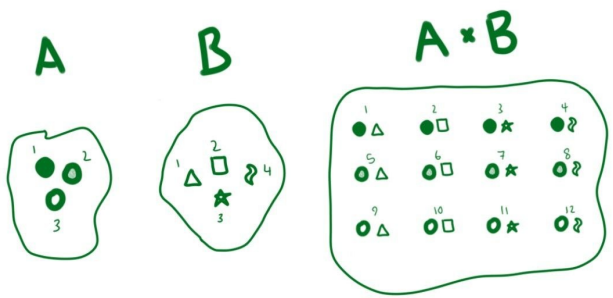
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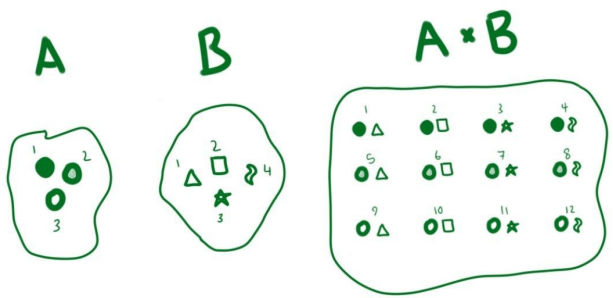
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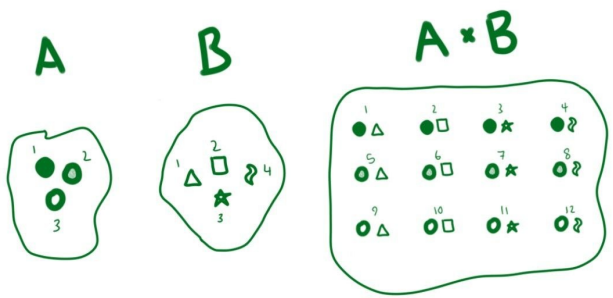


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$$f_{A \times B}(a, b) = (f_A(a) - 1) \cdot m + f_B(b).$$



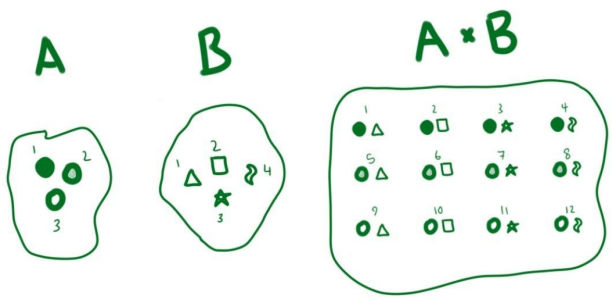
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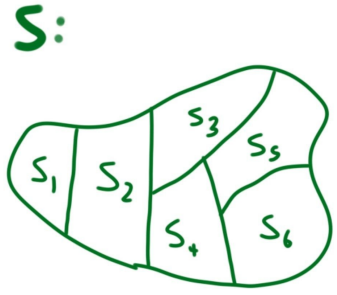
$$f_{A \times B}(a, b) = (f_A(a) - 1) \cdot m + f_B(b).$$

More generally, for sets $S_1, \ldots, S_k$, we have $|S_1 \times \cdots \times S_k| = |S_1| \cdots |S_k|$.



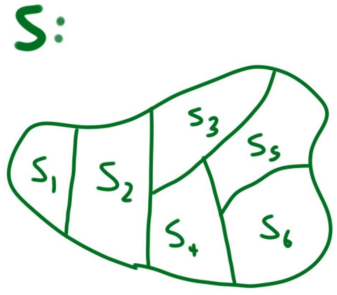
Size of a Partition

Definition: A partition of a set S is a collection of disjoint subsets $S_1, S_2, \ldots, S_k$ such that $S = S_1 \cup S_2 \cup \ldots \cup S_k$.



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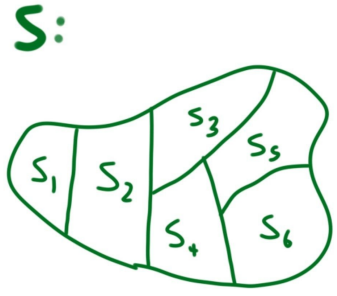
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Fact 2: If S is a finite set and $S_1, S_2, \ldots, S_k$ is a partition of S, then $|S| = |S_1| + |S_2| + \ldots + |S_k|$.

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Proof Idea: HW4 problem 2.

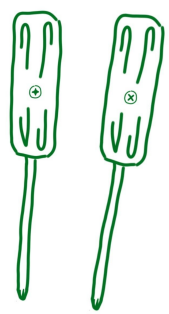
Two Facts About Set Sizes

Fact 1 (multiplication principle): If $S_1, S_2, \ldots, S_k$ are finite sets, then

$$|S_1 \times S_2 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdot \ldots \cdot |S_k|.$$

Fact 2 (addition principle): If S is a finite set and $S_1, S_2, \ldots, S_k$ is a partition of S, then

$$|S| = |S_1| + |S_2| + \ldots + |S_k|.$$



Counting

Example counting questions:

- How many ways are there to arrange the letters in the word "MISSISSIPPI"?
- How many ways are there to choose 5 cards from a standard deck of 52 cards?
- How many different outfits can you make if you have 13 shirts, 12 pairs of pants, and 5 dresses?



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These are secretly questions about sizes of sets.
e.g. "What is $|\{\text{arrangements of letters in MISSISSIPPI}\}|$?"



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26 options for the first, 25 options for the second, 24 options for the third, and so on.



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Example: The permutations of the set $\{1, 2, 3\}$ are $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$.

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Fact: The number of permutations of a set of size n is $n!$.

Counting Permutations

Question 2: There are 26 people in a room. How many ways are there to select and arrange 5 of them in a line?





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26 options for the first, 25 options for the second, . . . , and 22 options for the fifth.





Counting Choices

Question 3: There are 26 people in a room. How many ways are there to choose 5 of them?





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Answer: $\frac{26!}{21!5!}$

There are $\frac{26!}{21!}$ ways to arrange 5 people in a line, but each group of 5 people is overcounted 5! times.





Binomial Coefficients

Definition: We define the following notation:

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This is also called a binomial coefficient.



Multiplication Principle

Theorem: Suppose I can uniquely specify every element of A with a sequence of k choices, where each choice i has n_i options. Then, $|A| = n_1 \cdot n_2 \cdot \ldots \cdot n_k$.



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Proof Idea: Suppose S_i is the set of options for choice i. The sequence creates a bijection between A and $S_1 \times S_2 \times \ldots \times S_k$, so

$$|A| = |S_1 \times S_2 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdot \ldots \cdot |S_k|$$

Then the result follows from the multiplication principle.



Exercises

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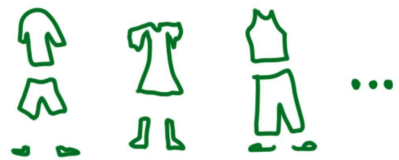


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- a shirt, pants, and shoes, or
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By the multiplication principle, there are $a \cdot b \cdot d$ ways to make the first type of outfit, and $c \cdot d$ ways to make the second type of outfit.

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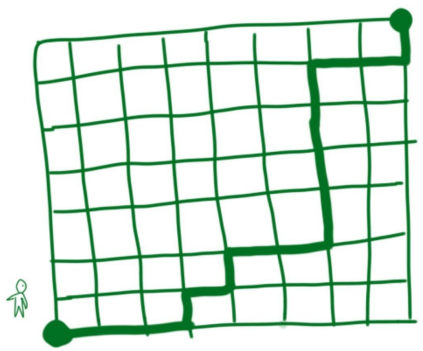
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- a shirt, pants, and shoes, or
- a dress and shoes.

By the multiplication principle, there are $a \cdot b \cdot d$ ways to make the first type of outfit, and $c \cdot d$ ways to make the second type of outfit.By the addition principle, there are $abd + cd$ total outfits.

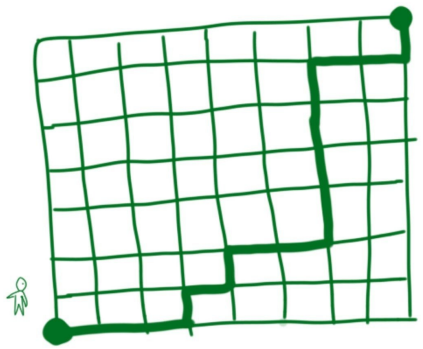
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Exercise 2: How many ways are there to go from the bottom left corner to the top right corner of a $m \times n$ grid, if you can only move up or right?



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Solution: $\binom{m+n}{m}$. Each path consists of m moves up and n moves right, for a total of $m+n$ moves. The number of ways to choose which m of the $m+n$ moves are up is $\binom{m+n}{m}$. This choice uniquely specifies the path and every path can be specified in this way.

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Solution: $\binom{11}{4} \cdot \binom{7}{4} \cdot \binom{3}{2} \cdot \binom{1}{1}$. We can uniquely specify any arrangement of the letters by choosing which 4 of the 11 positions are M's, then which 4 of the remaining 7 positions are I's, then which 2 of the remaining 3 positions are S's, and finally which of the remaining 1 position is P.

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Alt. Solution Idea: $\frac{11!}{4!4!2!1!}$. There are 11! ways to arrange the letters if they were all distinct, but we have overcounted by a factor of 4! for the M's, 4! for the I's, 2! for the S's, and 1! for the P's.

Exercises

Exercise 3: How many ways are there to choose 5 cards from a standard deck of 52 cards such that they are not all the same suit?



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Solution: $\binom{52}{5} - 4 \cdot \binom{13}{5}$. There are $\binom{52}{5}$ ways to choose 5 cards from a standard deck of 52 cards. There are $4 \cdot \binom{13}{5}$ ways to choose 5 cards that are all the same suit, $\binom{13}{5}$ for each of the 4 suits.

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Solution: $\binom{52}{5} - 4 \cdot \binom{13}{5}$. There are $\binom{52}{5}$ ways to choose 5 cards from a standard deck of 52 cards. There are $4 \cdot \binom{13}{5}$ ways to choose 5 cards that are all the same suit, $\binom{13}{5}$ for each of the 4 suits. By the addition principle, there are $\binom{52}{5} - 4 \cdot \binom{13}{5}$ ways to choose 5 cards that are not all the same suit.

Counting Techniques

Counting techniques so far:

- Uniquely specify each element with a sequence of choices and use the multiplication principle.
- Partition the set into disjoint subsets and use the addition principle.
- Overcount and divide by the number of times each element is counted.
- Overcount and subtract the number of elements that are counted too many times.

