Start of Lecture 10

Last time: Countability

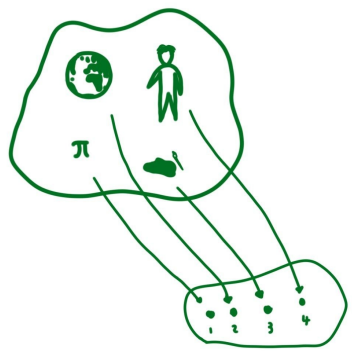
Today: Uncountability, Cardinality, Pigeonhole Principle



Finite Sets: Recap

Definition: A set A is finite if there exists some $n \geq 0$ and a bijection

$$f : A \to [n].$$



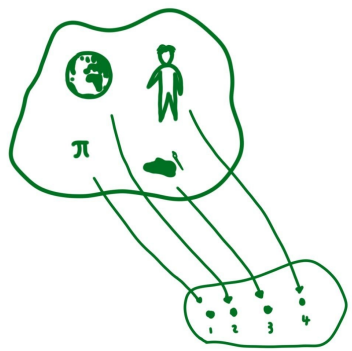
Finite Sets: Recap

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In this case, we say that A has size n and write

$$|A| = n.$$



Countability: Recap



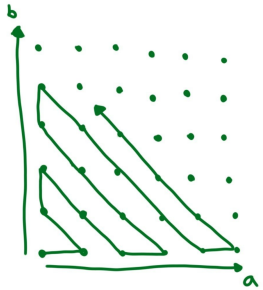
Definition:
A set S is countably infinite if there is a bijection $\mathbb{N} \to S$.

Example: $\mathbb{Z}$ is countably infinite.

Countability: Recap

Recall:

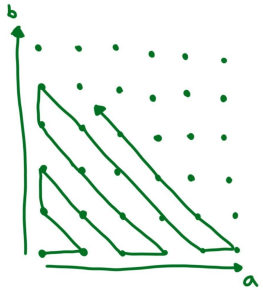
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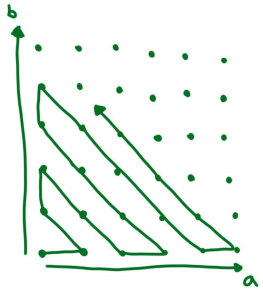
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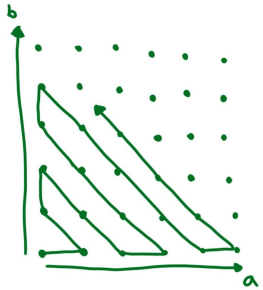
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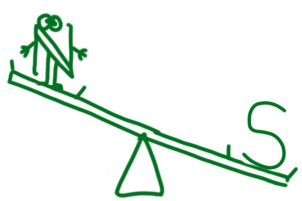
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Each of these sets can be arranged into an infinite list.

Uncountability

Definition: If a set S is infinite and not countably infinite, we call it uncountable.



Uncountability

Theorem: The set of infinite binary sequences

$$S := \{(b_1, b_2, b_3, \ldots) : b_i \in \{0, 1\}\}$$

is uncountable.



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Proof: Suppose, for contradiction, that we could list every infinite binary sequence:

$$\begin{aligned} s_1 &= (b_{11}, b_{12}, b_{13}, \ldots) \\ s_2 &= (b_{21}, b_{22}, b_{23}, \ldots) \\ s_3 &= (b_{31}, b_{32}, b_{33}, \ldots) \\ &\vdots \end{aligned}$$



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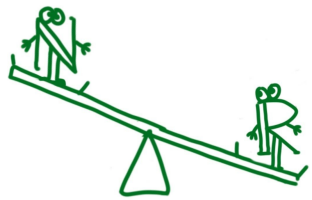
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Define $t = (t_1, t_2, t_3, \ldots)$ by $t_n = 1 - b_{nn}$. Then $t \neq s_n$ for every n, since their nth bits differ. Thus t is missing from the list, a contradiction. Therefore S is uncountable. □



Uncountability

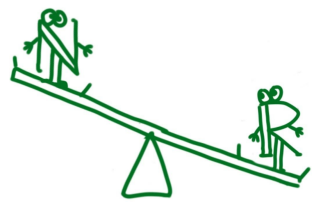
Exercise: Prove that $\mathbb{R}$ is uncountable.



Uncountability

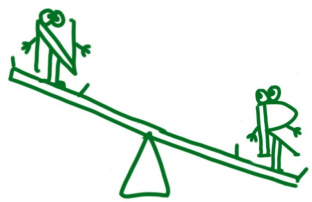
Exercise: Prove that $\mathbb{R}$ is uncountable.

Hint: Every real number has a decimal expansion.



Uncountability

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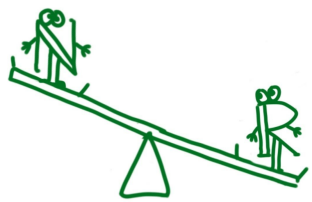


Uncountability

Theorem: $\mathbb{R}$ is uncountable.

Proof: Suppose, for contradiction, that we could list every number in $\mathbb{R}$:

$$\begin{aligned} x_1 &= \cdots . d_{11} d_{12} d_{13} \cdots \\ x_2 &= \cdots . d_{21} d_{22} d_{23} \cdots \\ x_3 &= \cdots . d_{31} d_{32} d_{33} \cdots \\ &\vdots \end{aligned}$$

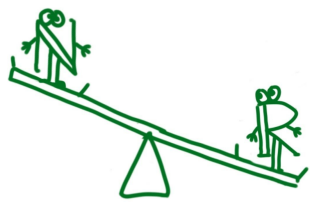


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Define $y = 0.c_1 c_2 c_3 \cdots$, where

$$c_n = \begin{cases} 1 & \text{if } d_{nn} \neq 1, \\ 2 & \text{if } d_{nn} = 1. \end{cases}$$

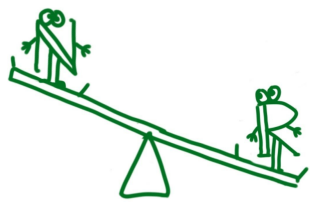
Then $y \in (0, 1)$, but $y \neq x_n$ for every n because their nth digits differ. Thus y is missing from the list, a contradiction. Therefore $\mathbb{R}$, is uncountable. $\square$

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This is called a diagonalization argument.

Uncountability

Theorem: The set $\mathcal{P}(\mathbb{N})$ is uncountable.



Uncountability



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Proof: Suppose, for contradiction, that we could list every subset of $\mathbb{N}$:

$$S_1, S_2, S_3, \ldots$$

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Uncountability



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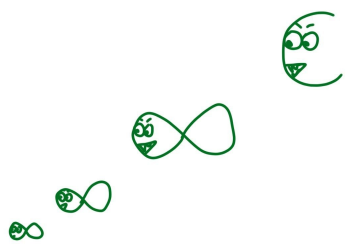
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Cardinalities

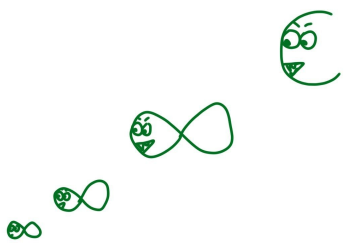
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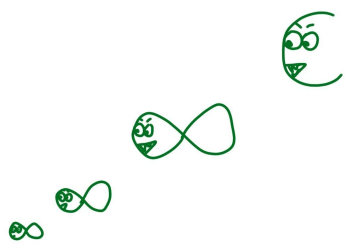


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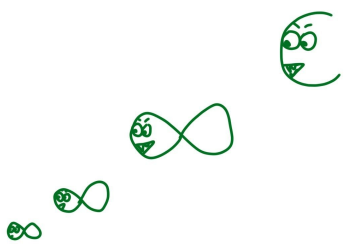
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Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$?



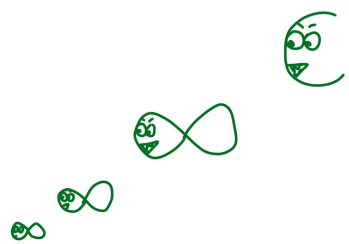
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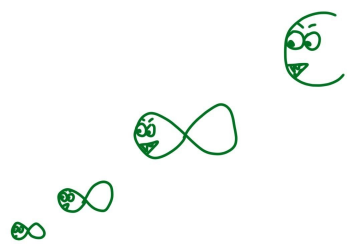
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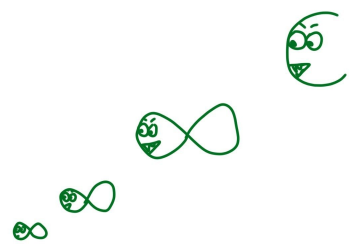
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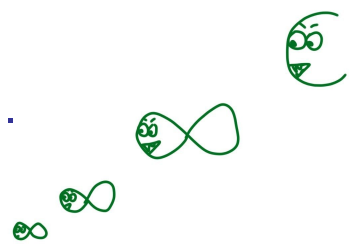
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Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$? Depends on axioms!

Is there a set S such that $|\mathbb{R}| < |S|$? Yes, $\mathcal{P}(\mathbb{R})$.
In fact, $|S| < |\mathcal{P}(S)|$ for any set S (we won't prove this).



Cardinalities

Exercises:

- Do $\mathbb{N}$ and $(0, 1)$ have the same cardinality?
- Do $(0, 1)$ and $(0, 1]$ have the same cardinality?



This slide is how far we got in lecture.

Properties of Countable Sets

Definition: A set is countable if it is finite or countably infinite.



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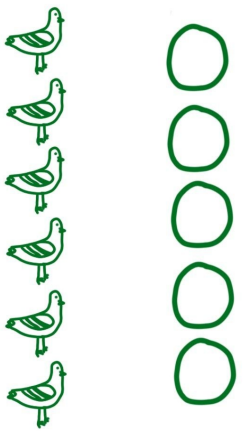
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Pigeonhole Principle

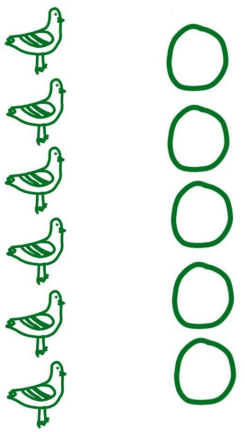
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Pigeonhole Principle: Let A and B be finite sets. If $|A| > |B|$, then no function $f : A \to B$ is injective.



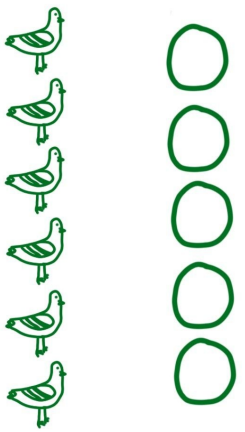
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Equivalently, for every $f : A \to B$, there exist distinct $a_1, a_2 \in A$ such that

$$f(a_1) = f(a_2).$$



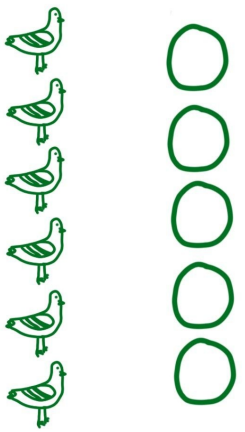
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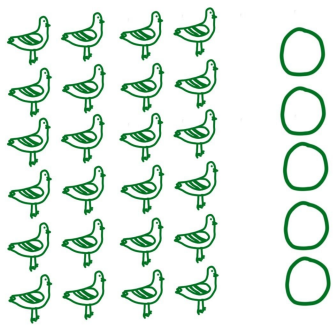


Example: If 13 people are in a room, at least two of them have the same birth month.

Pigeonhole Principle: Generalized version

Generalized Pigeonhole Principle: Let A and B be finite sets. If $|A| > k|B|$, then for any $f : A \to B$, there exists some $b \in B$ such that

$$|f^{-1}(b)| > k.$$



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Example: If 25 people are in a room, at least three of them have the same birth month.

