Midterm 2: Extra Practice Problems

Inverses

Problem 1 Consider the ceiling function $f : \mathbb{R} \to \mathbb{Z}$ via $f(x) = \lceil x \rceil$, which rounds a real number up to the nearest integer. That is, $f(x)$ is the smallest $z \in \mathbb{Z}$ where $x \leq z$.

Does f have a left inverse? If so, give a left inverse; if not, why not?

Does f have a right inverse? If so, give a right inverse, if not, why not?

Solution. No left inverse. A left inverse exists only if f is injective, and f is not injective: $f(0.5) = f(0.7) = 1$.

A right inverse does exist, since f is surjective — every $z \in \mathbb{Z}$ is $f(z) = \lceil z \rceil = z$. The inclusion $g : \mathbb{Z} \to \mathbb{R}$ via $g(z) = z$ is one: $f(g(z)) = \lceil z \rceil = z$ for all z.

Problem 2 The Collatz function is $c : \mathbb{N} \to \mathbb{N}$ via

$$c(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}.$$

Does c have a left inverse? If so, give a left inverse; if not, why not?

Does c have a right inverse? If so, give a right inverse, if not, why not?

Solution. No left inverse. c is not injective: $c(1) = 3(1) + 1 = 4$, and $c(8) = 8/2 = 4$.

A right inverse does exist, since c is surjective — every $n \in \mathbb{N}$ is reached from the even number $2n$, as $c(2n) = n$. So $g(n) = 2n$ is a right inverse: $c(g(n)) = c(2n) = n$.

Countability and Finiteness

Problem 3 Suppose that a set A is countable, and a set B is uncountable. Which of the following must be countable? Which must be uncountable?

(a) $A \setminus B$

(b) $B \setminus A$

(c) $A \times \mathbb{N}$

(d) $B \cap A$

(e) $B \cup \mathbb{N}$

(f) $B \setminus \mathbb{R}$

Solution. (a) Countable: $A \setminus B \subseteq A$, and a subset of a countable set is countable.

(b) Uncountable. Note $B \subseteq (B \setminus A) \cup A$, and if $B \setminus A$ were countable then B would be countable (since A is countable, and the union of two countable sets is countable).

(c) Countable: a product of two countable sets is countable.

(d) Countable: $B \cap A \subseteq A$.

(e) Uncountable: $B \cup \mathbb{N} \supseteq B$, and a superset of an uncountable set is uncountable.

(f) Can be either, depending on B. If $B \subseteq \mathbb{R}$, say $B = \mathbb{R}$ itself, then $B \setminus \mathbb{R} = \varnothing$, which is countable. But if no element of B is a real number, say $B = \mathcal{P}(\mathbb{N})$, then $B \setminus \mathbb{R} = B$, which is uncountable.

Problem 4 Consider the set $K = \{\text{“}a\text{”}, \text{“}aba\text{”}, \text{“}ababa\text{”}, \text{“}abababa\text{”}, ...\}$. Is K countable? Why or why not?

Solution. Countable. Every element of K is an a followed by some number of copies of ba, so

$$f : \mathbb{N} \to K \qquad \text{via} \qquad f(n) = a\underbrace{ba\,ba\cdots ba}_{n-1 \text{ copies}}$$

is a bijection: $f(n)$ has length $2n-1$, so distinct n give distinct strings, and every string in K has odd length $2n-1$ for exactly one $n \geq 1$ and equals $f(n)$. So K is countably infinite.

Problem 5 (a) Why must there be finitely many simple paths in a graph with finitely many vertices?

(b) Give an example of a graph G with finitely many vertices, and an infinite set of (not necessarily simple) paths on G.

Solution. (a) A simple path is determined by its sequence of vertices, and that sequence never repeats a vertex. With n vertices there are at most n choices for the first vertex, $n-1$ for the second, and so on, so there are at most

$$\sum_{k=1}^{n} \frac{n!}{(n-k)!}$$

such sequences in total. Not all of them are paths, but that is already a finite bound.

(b) Any graph with at least one edge works. Take $V = \{u, v\}$ with the single edge $\{u, v\}$: then

$$u, \quad u\,v, \quad u\,v\,u, \quad u\,v\,u\,v, \quad \ldots$$

are all paths, and there is one of every length, so there are infinitely many.

Problem 6 Prove that the interval $(-1, 1)$ is uncountable by giving a surjection to $\mathbb{R}$.

(Note: I changed this problem; it asked for a bijection $[-1, 1] \to \mathbb{R}$ before, which was more annoying to deal with.)

Solution. Define $f : (-1, 1) \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{x} - 1 & x > 0 \\ \frac{1}{x} + 1 & x < 0 \\ 0 & x = 0. \end{cases}$$

As x runs over $(0, 1)$, $\frac{1}{x}$ runs over $(1, \infty)$, so the first case covers $(0, \infty)$; as x runs over $(-1, 0)$, $\frac{1}{x}$ runs over $(-\infty, -1)$, so the second case covers $(-\infty, 0)$.

To check that f is surjective, take any $y \in \mathbb{R}$ and produce an $x \in (-1, 1)$ with $f(x) = y$.

- If $y = 0$, take $x = 0$: then $f(x) = 0 = y$.
- If $y > 0$, take $x = \frac{1}{y+1}$. Since $y + 1 > 1$ we have $0 < x < 1$, so $f(x) = \frac{1}{x} - 1 = (y + 1) - 1 = y$.
- If $y < 0$, take $x = \frac{1}{y-1}$. Since $y - 1 < -1$ we have $-1 < x < 0$, so $f(x) = \frac{1}{x} + 1 = (y - 1) + 1 = y$.

So every real number is $f(x)$ for some $x \in (-1, 1)$.

Now suppose $(-1, 1)$ were countable. A surjection from a countable set has a countable image, so $\mathbb{R}$ would be countable — but it isn't. So $(-1, 1)$ is uncountable.

(In fact f is a bijection: the three cases above hand back the *only* preimage of each y, so the rule $y \mapsto x$ is a two-sided inverse.)

Counting

Problem 7 A *straight* is a hand of 5 cards with sequential ranks (e.g., $4\heartsuit, 5\diamondsuit, 6\diamondsuit, 7\spadesuit, 8\clubsuit$). (The rank sequence is 2-10, J,Q,K,A, so $10\spadesuit, J\heartsuit, Q\heartsuit, K\diamondsuit, A\diamondsuit$ is also a straight).

A *flush* is a hand of 5 cards with all the same suit (e.g., $3\heartsuit, 5\heartsuit, 10\heartsuit, K\heartsuit, A\heartsuit$).

(a) How many straights are there?

(b) How many flushes are there?

(c) How many 5-card hands are both straights and flushes?

(d) How many 5-card hands are either a straight or a flush?

Solution. (a) A straight is fixed by choosing the lowest rank of the run and then a suit for each card. The five ranks must be consecutive, so the lowest one is 2 through 10, giving 9 runs, and then each of the 5 cards has 4 suits:

$$9 \cdot 4^5 = 9216.$$

(b) Choose the suit and then which 5 of its 13 ranks appear:

$$4 \cdot \binom{13}{5} = 4 \cdot 1287 = 5148.$$

(c) A hand that is both is a straight all of whose cards share a suit: 9 runs and 4 suits, so $9 \cdot 4 = 36$.

(d) By inclusion-exclusion,

$$9216 + 5148 - 36 = 14328.$$

Problem 8 A phone number is a sequence of 7 digits from 0 to 9.

(a) How many phone numbers are there?

(b) How many phone numbers are palindromes? A palindrome is a sequence that is the same when reversed, e.g. 1234321.

(c) How many phone numbers contain a 1?

(d) How many phone numbers contain a 1 or a 2?

Solution. (a) Each of the 7 digits is chosen independently, so $10^7 = 10{,}000{,}000$.

(b) A palindrome is determined by its first four digits: digits 5, 6, 7 must copy digits 3, 2, 1. So $10^4 = 10{,}000$.

(c) Count the complement. A number with no 1 has 9 choices at each digit, so there are 9^7 of them, and $10^7 - 9^7$ phone numbers with a 1.

Alternatively, we can count by partitioning over how many 1s. If there are $k \geq 1$ 1s, then there are $\binom{7}{k}$ spots for them and then 9^{7-k} spots for the other digits, so this would be

$$\sum_{k=1}^{7} \binom{7}{k} 9^{7-k}.$$

(d) Again the complement: a number containing neither a 1 nor a 2 has 8 choices at each digit, so there are $10^7 - 8^7$ phone numbers with a 1 or a 2.

Problem 9 I throw 5 darts at a dartboard. The dartboard has four sections: 0 points, 1 point, 10 points, and 100 points. How many possible scores can I end up with?

Solution. 56.

This is secretly stars and bars: each section is a "kid", and each dart is a "piece of candy": any assignment of darts to sections gives a distinct score. There are 5 stars and 3 bars, so this is

$$\binom{5+3}{3} = \binom{8}{3} = 56.$$

Problem 10 10 friends play a game of Blood on the Clocktower. There are 20 possible characters, of which 5 are evil and 15 are good. Every player gets assigned a distinct character. Exactly 3 players must get an evil character (and 7 must get a good character).

(a) How many ways can the characters be assigned?

(b) There is a pair of "linked" good characters: if anyone is assigned the Choirboy, then someone else must be assigned the King, and vice versa. How many possible ways can the characters be assigned now?

Solution. (a) Specify the assignment in three steps:

- choose which 3 of the 10 players are evil — $\binom{10}{3} = 120$ ways
- deal 3 distinct evil characters to those players — $5 \cdot 4 \cdot 3 = 60$ ways
- deal 7 distinct good characters to the other players — $15 \cdot 14 \cdots 9$ ways

By the multiplication principle, there are

$$\binom{10}{3} \cdot \frac{5!}{2!} \cdot \frac{15!}{8!} \text{ ways.}$$

Alternatively, you could also specify the assignment via:

- choose which 3 of the 10 players are evil — $\binom{10}{3} = 120$ ways
- choose 3 distinct evil characters — $\binom{5}{3}$ ways
- choose which of the evil players get which evil characters — 3! ways
- choose 7 distinct good characters — $\binom{15}{7}$ ways
- choose which of the good players get which good characters — 7! ways

This gets the same answer.

(b) We split into two cases: either both Choirboy and King are in the game, or neither. If both:

- choose which 3 of the 10 players are evil — $\binom{10}{3} = 120$ ways
- choose 3 distinct evil characters — $\binom{5}{3}$ ways

- choose which of the evil players get which evil characters — 3! ways
- Choose 7 distinct good characters. there are two cases:
  1. King and Choirboy are in the game, and we must choose 5 more. Pick 5 distinct good characters besides K&C — $\binom{13}{5}$ ways
  2. King and Choirboy are not in the game, and we must choose 7. Pick 7 distinct good characters (not K&C) — $\binom{13}{7}$ ways.
- choose which of the good players get which good characters — 7! ways

By addition principle, the 4th step has $\binom{13}{5} + \binom{13}{7}$ ways, so overall by multiplication principle there are

$$\binom{10}{3} \cdot \binom{5}{3} \cdot 3! \cdot \left(\binom{13}{5} + \binom{13}{7} \right) \cdot 7! \text{ ways}$$

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Problem 11 Using counting in two ways, prove that

$$\binom{2n}{3} = 2\binom{n}{3} + 2\binom{n}{1}\binom{n}{2}$$

Solution. Both sides count the following set. Suppose I have n friends and n enemies, and I want to choose a team of 3 people.

LHS: There are $2n$ people, and I'm choosing a team of 3, so there are $\binom{2n}{3}$ ways.

RHS: Do casework on how many of the people are my enemies.

1. All 3 people on the team are my friends. There are $\binom{n}{3}$ ways to do this.
2. 2 people are my friends and one is my enemy. This is a 2-step process: first, choose 2 friends $\binom{n}{2}$, and then choose 1 enemy $\binom{n}{1}$.
3. 2 people are my enemies and one is my friend. This is a 2-step process: first, choose 2 enemies $\binom{n}{2}$, and then choose 1 friend $\binom{n}{1}$.
4. All 3 people on the team are my enemies. There are $\binom{n}{3}$ ways to do this.

These cases cover the whole set and are disjoint, so by the addition principle, there are $\binom{n}{3} + \binom{n}{2}\binom{n}{1} + \binom{n}{2}\binom{n}{1} + \binom{n}{3}$ ways, which is equal to the RHS.

Thus by counting in two ways, LHS=RHS.

ALTERNATIVE (more boring) SOLUTION

Both sides count the 3-element subsets of a $2n$-element set. Split that set into two halves A and B, each of size n.

Counting directly, there are $\binom{2n}{3}$ such subsets.

Counting by how many elements come from each half — the four cases partition the subsets, so we add:

- all three from A $\binom{n}{3}$
- all three from B $\binom{n}{3}$
- one from A, two from B $\binom{n}{1}\binom{n}{2}$
- two from A, one from B $\binom{n}{2}\binom{n}{1}$

which totals $2\binom{n}{3} + 2\binom{n}{1}\binom{n}{2}$. The two counts are of the same set, so they are equal.

Problem 12 Using counting in two ways, prove that

$$n^3 = (n-1)^3 + \binom{3}{1}(n-1)^2 + \binom{3}{2}(n-1) + 1$$

Solution.

Both sides count the following set. I have n cats (labeled $1, 2, \cdots, n$), and I want to put a hat on one, socks on another, and a scarf on another. (One cat can get multiple items).

LHS: This is a 3-step process:

- Choose a cat to get the hat n ways
- Choose a cat to get the scarf n ways
- Choose a cat to get the socks n ways

By the Multiplication Principle, there are n^3 ways.

RHS: We do casework on how many items of clothing cat #1 gets.

- If cat 1 gets no clothing, then all 3 items must go to cats among the other $n-1$. This is a 3-step process, with $n-1$ options each time, so there are $(n-1)^3$ ways total.
- If cat 1 gets one item, then this is a two step process:
  - Choose which item cat 1 gets $\binom{3}{1}$ ways
  - Choose which of the $n-1$ cats get the other two items $(n-1)^2$ ways.
- If cat 1 gets two items, this is also a 2-step process:
  - Choose which 2 items cat 1 gets $\binom{3}{2}$ ways
  - Choose which of the $n-1$ cats get the remaining item $n-1$ ways.
- If cat 1 gets all 3 items, there is only one way to do this.

Since this is a partition (the cases are disjoint and cover all cases), by the addition principle, there are $(n-1)^3 + \binom{3}{1}(n-1)^2 + \binom{3}{2}(n-1) + 1$ ways total. By counting in two ways, LHS=RHS.

ALTERNATIVE (more boring) PROOF

Both sides count the triples (x, y, z) with $x, y, z \in [n]$.

Counting directly, there are $n \cdot n \cdot n = n^3$ of them.

Counting by how many coordinates are equal to n — again the cases partition the triples:

- no coordinate is n: each is one of the other $n-1$ values $(n-1)^3$
- exactly one is n: choose which coordinate, the others are among $n-1$ values $\binom{3}{1}(n-1)^2$
- exactly two are n: choose which two, the last is among $n-1$ values $\binom{3}{2}(n-1)$
- all three are n 1

Adding gives the right-hand side, and both count the same set.

Graphs

Problem 13 Recall that the star graph on n vertices has one central vertex and $n-1$ "outer" vertices, each with one edge connecting to the central vertex.

(a) How many induced subgraphs does the n-vertex star graph have?

(b) Does the n-vertex star graph have any disconnected induced subgraphs? If so, how many? If not, why not?

Solution. (a) 2^n. An induced subgraph is determined by the set of vertices you keep — the edges are then forced — and different vertex sets give different subgraphs. So the induced subgraphs correspond to the subsets of V, of which there are 2^n.

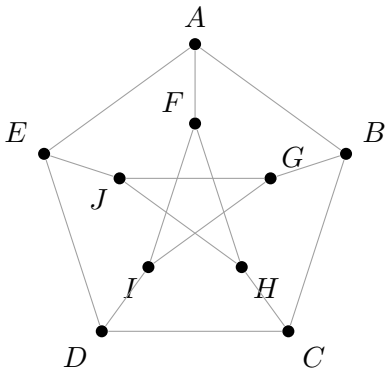
(b) Yes, $2^{n-1} - n$ of them.

Any induced subgraph that contains the central vertex will be connected. So, the disconnected induced subgraphs must have only "outside" vertices. And, it must have at least 2 vertices (if it only has 1, then it is connected).

There are 2^{n-1} subsets of the "outside" vertices. 1 of them is empty, and $n-1$ more of them contain only one vertex. So, there are a total of

$$2^{n-1} - 1 - (n-1) = 2^{n-1} - n \text{ disconnected induced subgraphs of the star}$$

Problem 14 This graph is called the "Petersen graph":



(a) Does the Petersen graph have a simple path with 10 vertices?

(b) Is the Petersen graph bipartite? If so, give a bipartition. If not, explain why not.

(c) Give a 4-independent set of the Petersen graph.

(d) Using the fact that the Petersen graph is 3-regular, explain why the Petersen graph cannot have a 6-independent set.

(Hint: suppose a 6-I.S., V', exists. All the edges adjacent to V' must go to vertices $V \setminus V'$, which has 4 nodes.) Why is this impossible?

Solution. (a) Yes. One is

$$A,\ B,\ C,\ D,\ E,\ J,\ H,\ F,\ I,\ G.$$

(b) No. The outer pentagon A, B, C, D, E, A is a cycle of length 5, which is odd, and a bipartite graph has no odd cycles. (If the graph was partitioned into sets X and Y, A and B would have to be in different sets, B and C would as well, C and D would, D and E would and so A and E would be in the same set. But they have an edge, so the graph isn't bipartite.)

(c) $\{A, C, I, J\}$ works. The neighbors of A are B, E, F; of C are B, D, H; of I are D, F, G; of J are E, G, H. No one of the four appears in another's list, so no two are adjacent.

(d) Suppose V' were an independent set with $|V'| = 6$, leaving 4 vertices outside it. No edge joins two vertices of V', so every edge at a vertex of V' has its other end in those 4 vertices. Each vertex of V' has degree 3, so that is $6 \cdot 3 = 18$ edge-ends landing on $V \setminus V'$. But those 4 vertices have only $4 \cdot 3 = 12$ edge-ends in total, and $18 > 12$. So no such V' exists.

Problem 15 Prove that if G is connected, then there exists a cycle (not necessarily simple) that visits every vertex in G.

Solution. Let the vertices be $v_1, v_2, \ldots, v_n$, and assume $n \geq 2$ (a single vertex with no edges has no cycle at all).

Since G is connected, there is a path P_i from v_i to v_{i+1} for each $i < n$, and a path P_n from v_n back to v_1. Each P_{i+1} starts where P_i ends, so stringing them together,

$$P_1 P_2 \cdots P_n,$$

is itself a path. It starts and ends at v_1 and has length at least 1, so it is a cycle, and it visits every vertex, since v_i appears in P_i.

Note that this cycle will usually repeat vertices and edges — which is allowed, since we only need a cycle, not a simple one.

Problem 16 Using a longest simple path argument, prove that if every vertex in G has degree at least k, then G has a simple path with at least k vertices.

Solution. Since G is finite it has only finitely many simple paths, so we may take a longest one, $v_1, v_2, \ldots, v_m$.

Every neighbor of v_1 lies on this path: if v_1 had a neighbor w off the path, then $w, v_1, v_2, \ldots, v_m$ would be a longer simple path, contradicting our choice.

So all of v_1's neighbors are among $v_2, \ldots, v_m$. There are at least k of them and only $m - 1$ vertices available, so $m - 1 \geq k$. The path therefore has $m \geq k + 1 \geq k$ vertices.