Midterm 2: Extra Practice Problems

Inverses

Problem 1 Consider the ceiling function $f : \mathbb{R} \to \mathbb{Z}$ via $f(x) = \lceil x \rceil$, which rounds a real number up to the nearest integer. That is, $f(x)$ is the smallest $z \in \mathbb{Z}$ where $x \leq z$.

Does f have a left inverse? If so, give a left inverse; if not, why not?

Does f have a right inverse? If so, give a right inverse, if not, why not?

Problem 2 The Collatz function is $c : \mathbb{N} \to \mathbb{N}$ via

$$c(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}.$$

Does c have a left inverse? If so, give a left inverse; if not, why not?

Does c have a right inverse? If so, give a right inverse, if not, why not?

Countability and Finiteness

Problem 3 Suppose that a set A is countable, and a set B is uncountable. Which of the following must be countable? Which must be uncountable?

(a) $A \setminus B$

(b) $B \setminus A$

(c) $A \times \mathbb{N}$

(d) $B \cap A$

(e) $B \cup \mathbb{N}$

(f) $B \setminus \mathbb{R}$

Problem 4 Consider the set $K = \{\text{“}a\text{”}, \text{“}aba\text{”}, \text{“}ababa\text{”}, \text{“}abababa\text{”}, ...\}$. Is K countable? Why or why not?

Problem 5 (a) Why must there be finitely many simple paths in a graph with finitely many vertices?

(b) Give an example of a graph G with finitely many vertices, and an infinite set of (not necessarily simple) paths on G.

Problem 6 Prove that the interval $(-1, 1)$ is uncountable by giving a surjection to $\mathbb{R}$.

(Note: I changed this problem; it asked for a bijection $[-1, 1] \to \mathbb{R}$ before, which was more annoying to deal with.)

Counting

Problem 7 A *straight* is a hand of 5 cards with sequential ranks (e.g., $4\heartsuit, 5\diamondsuit, 6\diamondsuit, 7\spadesuit, 8\clubsuit$). (The rank sequence is 2-10, J,Q,K,A, so $10\spadesuit, J\heartsuit, Q\heartsuit, K\diamondsuit, A\diamondsuit$ is also a straight).

A *flush* is a hand of 5 cards with all the same suit (e.g., $3\heartsuit, 5\heartsuit, 10\heartsuit, K\heartsuit, A\heartsuit$).

(a) How many straights are there?

(b) How many flushes are there?

(c) How many 5-card hands are both straights and flushes?

(d) How many 5-card hands are either a straight or a flush?

Problem 8 A phone number is a sequence of 7 digits from 0 to 9.

(a) How many phone numbers are there?

(b) How many phone numbers are palindromes? A palindrome is a sequence that is the same when reversed, e.g. 1234321.

(c) How many phone numbers contain a 1?

(d) How many phone numbers contain a 1 or a 2?

Problem 9 I throw 5 darts at a dartboard. The dartboard has four sections: 0 points, 1 point, 10 points, and 100 points. How many possible scores can I end up with?

Problem 10 10 friends play a game of Blood on the Clocktower. There are 20 possible characters, of which 5 are evil and 15 are good. Every player gets assigned a distinct character. Exactly 3 players must get an evil character (and 7 must get a good character).

(a) How many ways can the characters be assigned?

(b) There is a pair of "linked" good characters: if anyone is assigned the Choirboy, then someone else must be assigned the King, and vice versa. How many possible ways can the characters be assigned now?

Problem 11 Using counting in two ways, prove that

$$\binom{2n}{3} = 2\binom{n}{3} + 2\binom{n}{1}\binom{n}{2}$$

Problem 12 Using counting in two ways, prove that

$$n^3 = (n-1)^3 + \binom{3}{1}(n-1)^2 + \binom{3}{2}(n-1) + 1$$

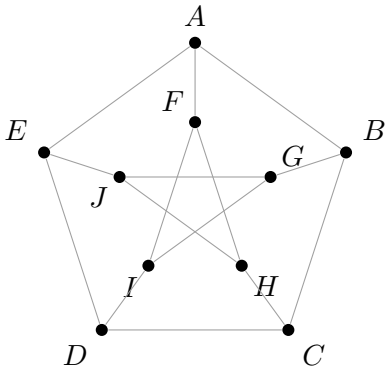
Graphs

Problem 13 Recall that the star graph on n vertices has one central vertex and $n-1$ "outer" vertices, each with one edge connecting to the central vertex.

(a) How many induced subgraphs does the n-vertex star graph have?

(b) Does the n-vertex star graph have any disconnected induced subgraphs? If so, how many? If not, why not?

Problem 14 This graph is called the "Petersen graph":



(a) Does the Petersen graph have a simple path with 10 vertices?

(b) Is the Petersen graph bipartite? If so, give a bipartition. If not, explain why not.

(c) Give a 4-independent set of the Petersen graph.

(d) Using the fact that the Petersen graph is 3-regular, explain why the Petersen graph cannot have a 6-independent set.

(Hint: suppose a 6-I.S., V', exists. All the edges adjacent to V' must go to vertices $V \setminus V'$, which has 4 nodes.) Why is this impossible?

Problem 15 Prove that if G is connected, then there exists a cycle (not necessarily simple) that visits every vertex in G.

Problem 16 Using a longest simple path argument, prove that if every vertex in G has degree at least k, then G has a simple path with at least k vertices.