Midterm 1: Extra Practice Problems

Quantifiers and Negation

Problem 1. Exactly one of the following is true:

$$\text{(i) } \forall x \in \mathbb{R}\, \exists y \in \mathbb{R},\ (x < y), \qquad \text{(ii) } \exists y \in \mathbb{R}\, \forall x \in \mathbb{R},\ (x < y).$$

(a) Which one is true? Prove it.

(b) Write the negation of the false one *without* using "not." Prove the negation.

Problem 2. Let P be the set of primes. Write the following in mathematical notation, then negate it without using "not": *every integer greater than* 1 *has a prime divisor.*

Problem 3. Let p be the statement $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m < n$. Write the negation of p without using "not," and prove whichever of p, $\neg p$ is true.

Induction

Problem 4. Prove that $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Problem 5. Prove that $6 \mid n^3 - n$ for all $n \in \mathbb{N}$.

Sets

Problem 6. For sets A, B, C, prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Problem 7. Prove or disprove: for all sets A, B, C, $(A \setminus B) \setminus C = A \setminus (B \setminus C)$.

Problem 8. Prove that $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Functions

Problem 9. Which of these are well-defined functions? For those that are not, say which property fails.

(a) $f : \mathbb{Z} \to \mathbb{N}$ via $f(n) =$ the number of divisors of n.

(b) $g : \mathbb{N} \to \mathbb{N}$ via $g(n) =$ the largest prime dividing n.

(c) $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ via $h(S) = \{2s : s \in S\}$.

Problem 10. Let $f : \mathbb{Z} \to \mathbb{Z}$ via $f(n) = n^2 + n$. Is f injective? Is it surjective? Prove your answers.

Problem 11. Let $f : A \to B$ and let $U, V \subseteq A$.

(a) Prove that $f[U \cup V] = f[U] \cup f[V]$.

(b) Prove that $f[U \cap V] \subseteq f[U] \cap f[V]$, and give an example where the containment is strict.

Working With a New Definition

Problem 12. Call a set $A \subseteq \mathbb{Z}$ *closed* iff for all $x, y \in A$, $x + y \in A$. Let

$$C = \{A \subseteq \mathbb{Z} : A \text{ is closed}\}.$$

(a) Give three elements of C.

(b) Show that $\mathbb{Z} \setminus \{0\} \notin C$.

(c) Prove that if $A, B \in C$ then $A \cap B \in C$.

(d) Give a well-defined injective function $F : \mathbb{N} \to C$, and prove it is injective.