Midterm 1: Extra Practice Problems

Quantifiers and Negation

Problem 1. Exactly one of the following is true:

$$\text{(i) } \forall x \in \mathbb{R}\, \exists y \in \mathbb{R},\ (x < y), \qquad \text{(ii) } \exists y \in \mathbb{R}\, \forall x \in \mathbb{R},\ (x < y).$$

(a) Which one is true? Prove it.

(b) Write the negation of the false one *without* using "not." Prove the negation.

Solution. (a) (i) is true. Let $x \in \mathbb{R}$ be arbitrary and take $y = x + 1$; then $x < y$.

(b) (ii) is false. Its negation is $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ with $x \geq y$. Let y be arbitrary and take $x = y$.

The two statements differ only in the order of the quantifiers. In (ii) the *same* y must work for every x; in (i) the y may depend on x.

Problem 2. Let P be the set of primes. Write the following in mathematical notation, then negate it without using "not": *every integer greater than* 1 *has a prime divisor.*

Solution. $\forall n \in \mathbb{Z},\ \big(n > 1 \implies \exists p \in P,\ p \mid n\big)$.

Negation: $\exists n \in \mathbb{Z}$ such that $n > 1$ and $\forall p \in P,\ p \nmid n$.

Note that negating $p \implies q$ gives $p \wedge \neg q$.

Problem 3. Let p be the statement $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m < n$. Write the negation of p without using "not," and prove whichever of p, $\neg p$ is true.

Solution. $\neg p$ is $\exists n \in \mathbb{N}$ such that $\forall m \in \mathbb{N},\ m \geq n$.

$\neg p$ is true: take $n = 1$. Every $m \in \mathbb{N}$ satisfies $m \geq 1$.

Induction

Problem 4. Prove that $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Solution.

Proof by induction. Base case: $n = 1$. $\sum_{i=1}^{1} i \cdot i! = 1 \cdot 1! = 1$, and $2! - 1 = 1$.

Inductive step. Fix $k \in \mathbb{N}$ and assume $\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1$. Then

$$\sum_{i=1}^{k+1} i \cdot i! = \big((k+1)! - 1\big) + (k+1)(k+1)! = (k+1)!\big(1 + (k+1)\big) - 1 = (k+2)! - 1,$$

as needed. □

Problem 5. Prove that $6 \mid n^3 - n$ for all $n \in \mathbb{N}$.

Solution.

Proof by induction. Base case: $n = 1$. $1^3 - 1 = 0$, and $6 \mid 0$.

Inductive step. Fix k and assume $6 \mid k^3 - k$. Then

$$(k+1)^3 - (k+1) = (k^3 - k) + 3k^2 + 3k = (k^3 - k) + 3k(k+1).$$

The first term is divisible by 6 by the inductive hypothesis. For the second, $k(k+1)$ is a product of two consecutive integers, so it is even, say $k(k+1) = 2m$; then $3k(k+1) = 6m$. So the sum is divisible by 6. □

Sets

Problem 6. For sets A, B, C, prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Solution.

Proof. ($\subseteq$) Let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$, so $x \notin B$ and $x \notin C$. Hence $x \in A \setminus B$ and $x \in A \setminus C$.

($\supseteq$) Let $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in A$, $x \notin B$, and $x \notin C$, so $x \notin B \cup C$ and therefore $x \in A \setminus (B \cup C)$. □

Problem 7. Prove or disprove: for all sets A, B, C, $(A \setminus B) \setminus C = A \setminus (B \setminus C)$.

Solution. False. Take $A = B = C = \{1\}$. Then

$$(A \setminus B) \setminus C = \varnothing \setminus \{1\} = \varnothing, \qquad A \setminus (B \setminus C) = \{1\} \setminus \varnothing = \{1\}.$$

One containment does always hold: if $x \in (A \setminus B) \setminus C$ then $x \in A$ and $x \notin B$, so $x \notin B \setminus C$, giving $x \in A \setminus (B \setminus C)$.

Problem 8. Prove that $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Solution.

Proof ($\Longrightarrow$). Suppose $A \subseteq B$, and let $X \in \mathcal{P}(A)$. Then $X \subseteq A \subseteq B$, so $X \subseteq B$, i.e. $X \in \mathcal{P}(B)$. □

Proof ($\Longleftarrow$). Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \subseteq A$, we have $A \in \mathcal{P}(A)$, so $A \in \mathcal{P}(B)$, which says exactly that $A \subseteq B$. □

Functions

Problem 9. Which of these are well-defined functions? For those that are not, say which property fails.

(a) $f : \mathbb{Z} \to \mathbb{N}$ via $f(n) =$ the number of divisors of n.

(b) $g : \mathbb{N} \to \mathbb{N}$ via $g(n) =$ the largest prime dividing n.

(c) $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ via $h(S) = \{2s : s \in S\}$.

Solution. (a) Not well-defined: $f(0)$ is undefined, since every integer divides 0 and so there is no such natural number. No totality.

(b) Not well-defined: $g(1)$ is undefined, since no prime divides 1. No totality.

(c) Well-defined. *Totality:* $h(S)$ is described for every $S \subseteq \mathbb{N}$. *Existence:* every element $2s$ is a natural number, so $h(S) \subseteq \mathbb{N}$ and thus $h(S) \in \mathcal{P}(\mathbb{N})$. *Uniqueness:* $\{2s : s \in S\}$ is determined by S.

Problem 10. Let $f : \mathbb{Z} \to \mathbb{Z}$ via $f(n) = n^2 + n$. Is f injective? Is it surjective? Prove your answers.

Solution. Neither.

Not injective: $f(0) = 0$ and $f(-1) = 1 - 1 = 0$, but $0 \neq -1$.

Not surjective: nothing maps to 1. Indeed $f(n) = n(n+1)$ is a product of two consecutive integers, hence even, and 1 is odd.

Problem 11. Let $f : A \to B$ and let $U, V \subseteq A$.

(a) Prove that $f[U \cup V] = f[U] \cup f[V]$.

(b) Prove that $f[U \cap V] \subseteq f[U] \cap f[V]$, and give an example where the containment is strict.

Solution. (a) $b \in f[U \cup V]$ iff there is $a \in U \cup V$ with $f(a) = b$, iff there is such an a in U or there is such an a in V, iff $b \in f[U]$ or $b \in f[V]$.

(b) If $b \in f[U \cap V]$, pick $a \in U \cap V$ with $f(a) = b$. Since $a \in U$ we get $b \in f[U]$, and since $a \in V$ we get $b \in f[V]$.

For strictness take $A = \{1, 2\}$, $B = \{1\}$, $f(1) = f(2) = 1$, $U = \{1\}$, $V = \{2\}$. Then $U \cap V = \varnothing$ so $f[U \cap V] = \varnothing$, while $f[U] \cap f[V] = \{1\}$.

Working With a New Definition

Problem 12. Call a set $A \subseteq \mathbb{Z}$ *closed* iff for all $x, y \in A$, $x + y \in A$. Let

$$C = \{A \subseteq \mathbb{Z} : A \text{ is closed}\}.$$

(a) Give three elements of C.

(b) Show that $\mathbb{Z} \setminus \{0\} \notin C$.

(c) Prove that if $A, B \in C$ then $A \cap B \in C$.

(d) Give a well-defined injective function $F : \mathbb{N} \to C$, and prove it is injective.

Solution. (a) $\mathbb{Z}$, the even integers, $\{0\}$, $\mathbb{N}$, and $\varnothing$ (vacuously) are all closed.

(b) $1 \in \mathbb{Z} \setminus \{0\}$ and $-1 \in \mathbb{Z} \setminus \{0\}$, but $1 + (-1) = 0 \notin \mathbb{Z} \setminus \{0\}$.

(c) Let $x, y \in A \cap B$. Since $x, y \in A$ and A is closed, $x + y \in A$; since $x, y \in B$ and B is closed, $x + y \in B$. So $x + y \in A \cap B$.

(d) Let $F(n) = n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$. This lands in C: if $nk, nj \in n\mathbb{Z}$ then $nk + nj = n(k + j) \in n\mathbb{Z}$.

Injective: suppose $n \neq m$, and without loss of generality $n < m$. Then $n \in F(n)$. But $n \notin F(m)$: if $n = mk$ then $k \neq 0$ (as $n \geq 1$), so $|n| = |m||k| \geq m > n$, a contradiction. So $F(n) \neq F(m)$.