Recitation 8: Probability

Solutions

1 Probability spaces

Problem 1. I choose 3 numbers at random from [10] (allowing repeats). What is the probability space (Ω, P)? What is the probability that all 3 numbers that I choose are different?

Solution. Ω is $[10]^3$, and P is $P(w) = 1/1000$ for all w.

There are $\binom{10}{3} \cdot 3!$ ways to choose 3 that are all different, so the probability that they are all different is

$$P(\text{all different}) = \frac{\binom{10}{3} \cdot 3!}{|\Omega|} = \frac{\binom{10}{3} \cdot 3!}{10^3} = \frac{720}{1000} = 0.72.$$

2 Conditional probability and independence

Problem 2. I roll 3 dice. Can you give examples of events A and B that are:

(a) Mutually exclusive but not independent?

(b) Independent but not mutually exclusive?

(c) Neither?

(d) Both?

Solution. (examples)

(a) A = first roll is a 6, B = first roll is a 3.

$A \cap B = \varnothing$, so they are mutually exclusive. $P(A) = P(B) = 1/6$, but $P(A \cap B) = 0 \neq \frac{1}{6} \cdot \frac{1}{6} = P(A) \cdot P(B)$, so they are not independent.

(b) A = first roll is a 6, B = last roll is a 6.

$A \cap B$ includes 6 outcomes (of the form $(6, x, 6)$ for $x = 1, 2, 3, 4, 5, 6$), so they are not mutually exclusive.

$P(A) = 1/6, P(B) = 1/6, P(A \cap B) = |A \cap B|/|\Omega| = 6/6^3 = 1/36$, so they are independent.

(c) A = first roll is even, B = first roll is 6. $P(A) = 1/2, P(B) = 1/6$. B is a subset of A so they are not disjoint. But $P(A \cap B) = P(B) \neq P(B) \cdot P(A)$, so they are not independent.

(d) $A = \Omega, B = \varnothing$. $A \cap B = \varnothing$, so they are mutually exclusive, and $P(A \cap B) = 0 = 1 \cdot 0 = P(A) \cdot P(B)$.

Problem 3. In lecture, we stated and used the fact that, for events A and B,

$$P(\overline{A} \mid B) = 1 - P(A \mid B).$$

Prove this fact using the basic definitions of probability (and set operations).

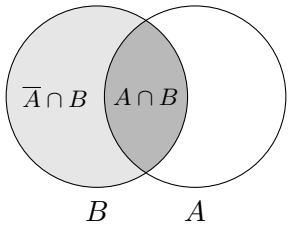
Solution.

Proof. By definition,

$$P(\overline{A} \mid B) = \frac{P(\overline{A} \cap B)}{P(B)},$$

so let's rewrite the numerator. Every outcome of B either is in A or is not, so B splits into two disjoint pieces:

$$B = (\overline{A} \cap B) \ \cup \ (A \cap B).$$



Since the two pieces are disjoint,

$$P(B) = P(\overline{A} \cap B) + P(A \cap B), \qquad \text{so} \qquad P(\overline{A} \cap B) = P(B) - P(A \cap B).$$

Plugging that back in,

$$P(\overline{A} \mid B) = \frac{P(B) - P(A \cap B)}{P(B)} = 1 - \frac{P(A \cap B)}{P(B)} = 1 - P(A \mid B). \qquad \square$$

Problem 4. There are three types of cars on the road, with different levels of reliability.

- 50% of cars are type 1. On any given day there is a 1/100 chance it breaks down.
- 30% of cars are type 2. On any given day there is a 1/200 chance it breaks down.
- 20% of cars are type 3. On any given day there is a 1/20 chance it breaks down.

A customer brings a broken-down car to the mechanic. What is the probability that it is a type 1 car? Type 2? Type 3?

Solution. By Bayes' theorem,

$$P(\text{type } x \mid \text{breaks down}) = \frac{P(\text{breaks down} \mid \text{type } x) \cdot P(\text{type } x)}{P(\text{breaks down})},$$

What is the denominator? The three types partition the cars, so by the law of total probability

$$\begin{aligned} P(\text{breaks down}) &= P(\text{breaks down} \mid \text{type 1})P(\text{type 1}) + P(\text{breaks down} \mid \text{type 2})P(\text{type 2}) \\ &\quad + P(\text{breaks down} \mid \text{type 3})P(\text{type 3}) \\ &= \tfrac{1}{100} \cdot 0.5 + \tfrac{1}{200} \cdot 0.3 + \tfrac{1}{20} \cdot 0.2 \\ &= 0.005 + 0.0015 + 0.01 = 0.0165. \end{aligned}$$

So

$$P(\text{type 1} \mid \text{breaks down}) = \frac{0.5 \cdot \frac{1}{100}}{0.0165} \approx 0.303,$$

$$P(\text{type 2} \mid \text{breaks down}) = \frac{0.3 \cdot \frac{1}{200}}{0.0165} \approx 0.091,$$

$$P(\text{type 3} \mid \text{breaks down}) = \frac{0.2 \cdot \frac{1}{20}}{0.0165} \approx 0.606.$$