Recitation 7: Eulerian paths, isomorphism

Solutions

1 Eulerian paths



Figure 1: Dominos laid out in a ring, where adjacent numbers match (thank you google images)

Example 1.1. A domino set has $\binom{7}{2}$ dominos, each of which has two sides with numbers from 0 to 6 (no repeats: no domino has the same number twice). Can you lay all of them out in a closed ring, so that touching ends match, as in Fig. 1?

Styopa's section did this in class. (We didn't have time to get to this in Gabriel's section). The domino problem is equivalent to finding an Eulerian cycle in K_7. We will go over this in recitation.

Problem 1. What if my domino set has sides that go from 0 to 5 instead?

Solution. The graph is K_6, which is 5-regular (every node has 5 neighbors). There is an Eulerian cycle if and only if every node has even degree, so there is no Eulerian cycle.

Problem 2. What if my domino set has sides that go from 0 to 5, but is missing the dominos $(0,1),(1,2),(2,3),(3,4),(4,5),(5,0)$?

Solution. The graph is 3-regular: every node lost two edges. There is still no Eulerian cycle, since there are vertices with odd degree.

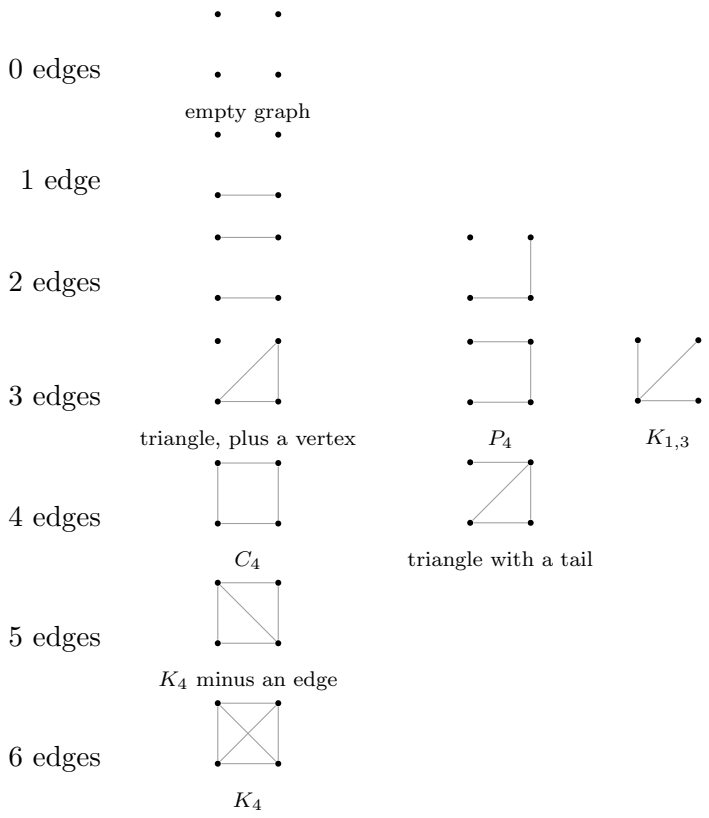
Problem 3. What if my domino set has sides that go from 0 to 5, but is missing the dominos $(0,1),(2,3),(4,5)$?

Solution. All vertices now have even degree, since we removed one edge from each vertex in K_6. So, there is an Eulerian cycle.

2 Isomorphism

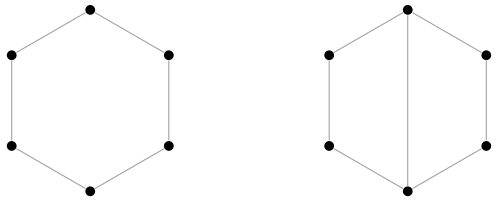
Problem 4. List all the different graphs on 4 vertices, where two graphs are "the same" if they are isomorphic.

Solution. There are 11. Organized by number of edges, with the same four vertex positions used throughout, so that only the edges change:



Problem 5. For each pair below, decide whether the two graphs are isomorphic. If they are, give the bijection. If they are not, say which invariant tells them apart.

(a)



G_1 G_2

Solution. Not isomorphic. G_1 has 6 edges and G_2 has 7, and isomorphic graphs have the same number of edges. (The degree sequences also differ: $2,2,2,2,2,2$ against $3,3,2,2,2,2$.)

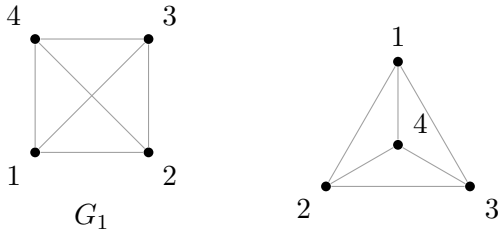
(b)



G_1 G_2

Solution. Not isomorphic. Both have 5 vertices and 6 edges, but $G1$ has a degree of vertex 4 and G_2 doesn't. (The degree sequences are $4,2,2,2,2$ for G_1 and $3,3,2,2,2$ for G_2).

(c)



G_1 G_2

Solution. Isomorphic. Both are K_4: in each one, all $\binom{4}{2} = 6$ pairs of vertices are joined by an edge. Since every pair is an edge in both graphs, *any* bijection is an isomorphism.

Problem 6. For each pair below, decide whether the two graphs are isomorphic. If they are, give the bijection. If they are not, give an invariant that tells them apart.

(a) Two trees:



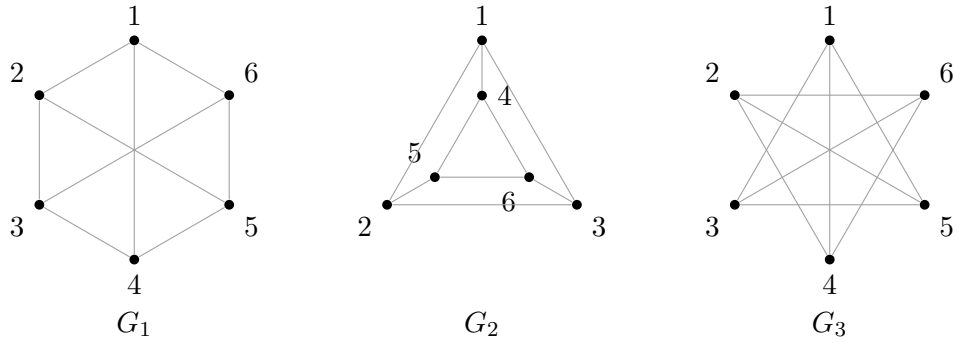
G_1 G_2

Solution. Not isomorphic. (Both are trees on 6 vertices with 5 edges, both have degree sequence $3, 2, 2, 1, 1, 1$, and both have diameter 4, so none of those invariants works.)

The degree-3 vertex of G_1 is adjacent to two leaves, and the degree-3 vertex of G_2 is adjacent to one.

An isomorphism sends a vertex to a vertex of the same degree, and sends its neighbors to the neighbors of its image, so the *degrees of the neighbors* of the degree-3 vertex must match. In G_1 those degrees are $2, 1, 1$; in G_2 they are $2, 2, 1$.

(b) Three graphs. Which pairs are isomorphic?



> G_3 is the complement of the 6-cycle $1, 2, 3, 4, 5, 6$.

Solution. G_2 and G_3 are isomorphic, and G_1 is isomorphic to neither.

All three have 6 vertices, 9 edges, and every vertex of degree 3, so the degree sequence is no help at all here.

G_1 has no triangle (induced 3-clique), whereas G_2 and G_3 have triangles.

G_2 and G_3 each have exactly two triangles, namely $1, 2, 3$ and $4, 5, 6$ in G_2, and $1, 3, 5$ and $2, 4, 6$ in G_3.

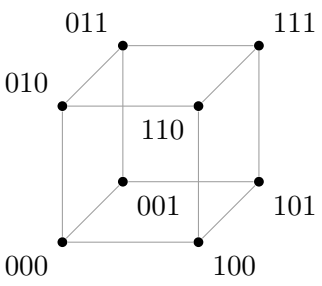
For $G_2 \cong G_3$: in G_3 the two triangles are $1, 3, 5$ and $2, 4, 6$, and the three remaining edges are 1–4, 2–5 and 3–6, which form a perfect matching between them. That is exactly how G_2 is built. Mapping $G_3 \to G_2$:

$$1 \mapsto 1, \quad 3 \mapsto 2, \quad 5 \mapsto 3, \quad 4 \mapsto 4, \quad 6 \mapsto 5, \quad 2 \mapsto 6.$$

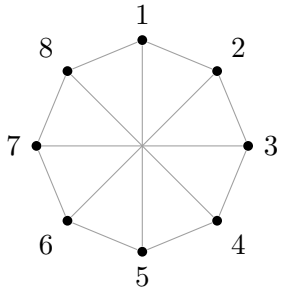
The triangle $1, 3, 5$ goes to the triangle $1, 2, 3$; the triangle $2, 4, 6$ goes to the triangle $6, 4, 5$; and the matching edges 1–4, 2–5, 3–6 go to 1–4, 6–3, 2–5.

(Intuitively, "turn" one of the triangles in G_3 180 degrees and make it a little bigger; this recovers G_2.)

(c) The cube Q_3, and an octagon with its four long diagonals:



Q_3



G_4

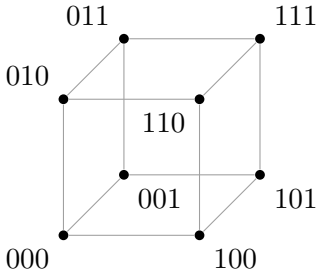
Solution. Not isomorphic.

Both have 8 vertices, 12 edges, and every vertex of degree 3. Both are also triangle-free, so counting triangles does not separate them either. However, Q_3 is bipartite (from the homework!) and G_4 is not, because it has a 5-cycle.

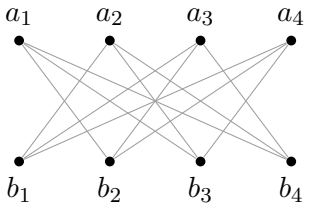
An isomorphism sends a cycle of length k to a cycle of length k, so one graph has an odd cycle and the other does not.

Counting 4-cycles also works: Q_3 has 6 of them and G_4 has 4.

(d) (hard) The cube Q_3 again, and $K_{4,4}$ with 4 edges deleted:



Q_3



G_5

> G_5 has an edge from a_i to b_j whenever $i \neq j$. So the only missing edges are a_1b_1, a_2b_2, a_3b_3 and a_4b_4.

Solution. Isomorphic.

Every invariant we have matches: both have 8 vertices, 12 edges, every degree 3, both are bipartite, both are triangle-free, and both have exactly six 4-cycles. So let's try to find the bijection.

The bipartiteness of Q_3 is helpful, since we know that a bipartition of Q_3 must correspond to the bipartition of G_4 into $\{a_1, a_2, a_3, a_4\}, \{b_1, b_2, b_3, b_4\}$.

Now take the four vertices of Q_3 whose labels have an even number of 1s, and call them a_1, a_2, a_3, a_4. Let b_i be the opposite of a_i:

a_1	a_2	a_3	a_4
000	011	101	110
b_1	b_2	b_3	b_4
111	100	010	001

Each b_i has an odd number of 1s, so this lists all 8 vertices.

Why is this an isomorphism? Every edge of Q_3 joins an even-weight vertex to an odd-weight one, so every edge runs from some a_i to some b_j. And a_i is adjacent to b_j exactly when their labels differ in one bit, which fails only when they differ in all three bits, which happens only when $j = i$. So $a_i \sim b_j$ if and only if $i \neq j$, which is precisely G_5.