Recitation 6: Counting in Two Ways, Graphs, and Pigeonhole
Solutions

You may leave answers unsimplified, e.g. in terms of $\binom{n}{k}$ and factorials.

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1 Counting in Two Ways

Problem 1. Using counting in two ways, show that

$$\binom{n}{3} = \sum_{i=1}^{n} (i-1)(n-i).$$

Solution. Both sides count the number of 3-element subsets of $[n]$.

Left side. By definition, there are $\binom{n}{3}$ of them.

Right side. Every 3-element subset has a unique middle element. Partition the subsets by what that middle element is. If the middle element is i, then we must choose the smallest element from $\{1, \ldots, i-1\}$, which can be done in $i-1$ ways, and the largest from $\{i+1, \ldots, n\}$, which can be done in $n-i$ ways. By the multiplication principle there are $(i-1)(n-i)$ subsets with middle element i.

Every 3-element subset is counted exactly once this way, so by the addition principle the total is $\sum_{i=1}^{n}(i-1)(n-i)$.

Since both sides count the same set, they are equal.

> The terms $i = 1$ and $i = n$ contribute 0: those are the cases where there is no room for a smaller (resp. larger) element.

Problem 2. Using counting in two ways, show that

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

Solution. Both sides count the number of $(k+1)$-element subsets of $\{0, 1, 2, \ldots, n\}$, a set with $n+1$ elements.

Right side. By definition, there are $\binom{n+1}{k+1}$ of them.

Left side. Every such subset has a unique largest element. Partition the subsets by what that largest element is. (This is a partition because every subset has a largest element, and no subset can have more than one largest element.) If the largest element is i, then the remaining k elements are chosen from $\{0, 1, \ldots, i-1\}$, a set of size i, so there are $\binom{i}{k}$ such subsets.

Summing over all possible largest elements i gives $\sum_{i=0}^{n} \binom{i}{k}$.

Since both sides count the same set, they are equal.

> Note that $\binom{i}{k} = 0$ when $i < k$, so the terms at the start of the sum vanish — a subset of size $k+1$ cannot have a largest element smaller than k. Starting the sum at $i = k$ gives the same answer.

2 Graph Theory

Problem 3. For each of the following sequences, either draw a graph whose vertices have those degrees, or prove that no such graph exists.

(a) $1, 2, 2, 3, 3, 4$

(b) $4, 4, 4, 4, 4$

(c) $1, 1, 1, 1, 1$

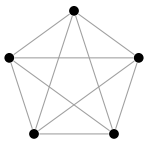
(d) $5, 3, 2, 2, 1, 1$

(e) $3, 3, 3, 3, 3$

(f) $5, 5, 1, 1, 1, 1$

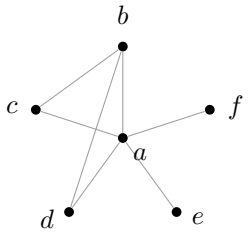
Solution. (a) **Impossible.** The degrees sum to 15, which is odd. By the handshake lemma the sum of the degrees is $2|E|$, which is even.

(b) **Possible:** this is K_5, in which every vertex is adjacent to the other four.



(c) **Impossible.** The degrees sum to 5, which is odd.

(d) **Possible.** Let a be adjacent to each of b, c, d, e, f; this gives a degree 5 and each of the others degree 1 so far. Now add the edges bc and bd, which brings b to degree 3 and c and d to degree 2.



(e) **Impossible.** The degrees sum to 15, which is odd. (More generally, a d-regular graph on n vertices needs nd to be even.)

(f) **Impossible** but not because of parity: the degrees sum to 14, which is even.

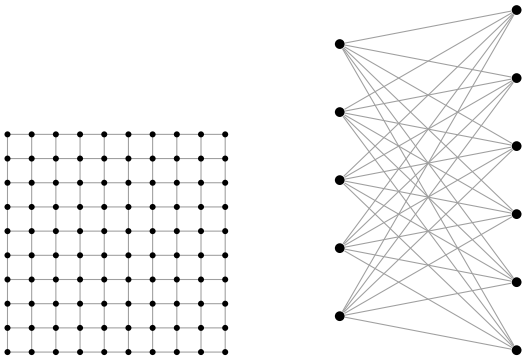
Suppose such a graph existed, and let u and v be the two vertices of degree 5. The graph has 6 vertices, so u is adjacent to all five other vertices, and so is v. Let w be any one of the four remaining vertices. Then w is adjacent to both u and v, so $\deg(w) \geq 2$. But the sequence requires w to have degree 1, a contradiction.

Problem 4.

(a) How many 3-cliques are there in K_n?

(b) What is the largest independent set in the 10×10 grid graph, drawn below? Prove that it is the largest.

(c) What is the largest independent set in $K_{5,6}$, drawn below? Prove that it is the largest.

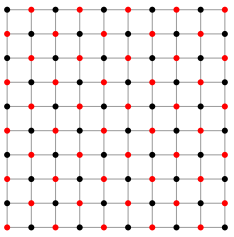


Solution. (a) A 3-clique is a set of three vertices that are pairwise adjacent. In K_n every pair of vertices is adjacent, so *every* 3-element set of vertices is a 3-clique. There are $\binom{n}{3}$ of them.

(b)

The answer is 50.

A set of size 50 exists. Colour the cell (x, y) black if $x + y$ is even and white if $x + y$ is odd, like a checkerboard. Two cells joined by an edge differ by 1 in exactly one coordinate, so their sums $x + y$ differ by 1 and have opposite parities. So no edge joins two black cells, and the 50 black cells form an independent set.



No larger set exists. Pair up the cells into 50 dominoes: in each row, pair columns 1–2, 3–4, 5–6, 7–8, 9–10. The two cells of a domino are adjacent, so an independent set contains at most one of them. There are 50 dominoes, so an independent set has at most 50 cells.

(c) The answer is 6: the side of size 6.

A set of size 6 exists. No two vertices on the same side of a complete bipartite graph are adjacent, so the side with 6 vertices is an independent set.

No larger set exists. Suppose S is an independent set containing a vertex u from the 5-side and a vertex v from the 6-side. In $K_{5,6}$ every vertex on one side is adjacent to every vertex on the other, so u and v are adjacent, contradicting independence. So S lies entirely within one side, and therefore $|S| \leq 6$.

3 Pigeonhole principle

Problem 5. Chris picks 51 numbers from [100]. Prove that no matter which numbers he picks, two of them sum to 101.

Solution. We apply the pigeonhole principle, where the 50 holes are the 50 pairs

$$\{1, 100\}, \{2, 99\}, \{3, 98\}, \cdots, \{50, 51\}$$

and the pigeons are whichever 51 numbers Chris chooses. by PHP, two pigeons must be in the same hole, so there's a pair that adds up to 101.

> An alternative proof, using the formal version of PHP with a non-injective function:

Let $S \subseteq [100]$ be Chris's set, so $|S| = 51$. Define

$$f : S \to [50] \qquad \text{via} \qquad f(x) = \min(x,\ 101 - x).$$

f is well defined. If $x \leq 50$ then $f(x) = x \in [50]$. If $x \geq 51$ then $101 - x$ lies between 1 and 50, so $f(x) = 101 - x \in [50]$. Either way $f(x) \in [50]$.

Applying the pigeonhole principle. We have $|S| = 51 > 50 = |[50]|$, so f is not injective: there are $x \neq y$ in S with $f(x) = f(y)$. Call this common value k.

Finishing. The only elements of [100] with $\min(z, 101 - z) = k$ are $z = k$ and $z = 101 - k$. So x and y both lie in $\{k, 101 - k\}$, and since $x \neq y$, one of them is k and the other is $101 - k$. Hence $x + y = 101$.