Recitation 5: Counting
Solutions

You may leave answers unsimplified, e.g. in terms of $\binom{n}{k}$ and factorials.

Problem 1. In poker, a *three of a kind* is a 5-card hand with three cards of one rank, and two more cards of two different ranks (different from each other, and from the rank of the triple). For example, $\{7\heartsuit, 7\spadesuit, 7\clubsuit, 2\diamondsuit, J\clubsuit\}$ is a three of a kind.

(a) How many three of a kind hands are there?

(b) In lecture we counted *two pair* hands, and had to divide by 2 at the end. Did you do something similar here? If so, why? If not, why not?

Solution.

> The way this solution is written is how you should write your HW solutions.

(a) We can specify a three of a kind hand by a sequence of choices:

1. choose the rank of the triple — 13 ways
2. choose which three of that rank's four suits appear — $\binom{4}{3}$ ways
3. choose the ranks of the two single cards — $\binom{12}{2}$ ways
4. choose a suit for the lower-ranked single card — 4 ways
5. choose a suit for the higher-ranked single card — 4 ways

By the multiplication principle, the number of such hands is

$$13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4 \cdot 4 = 13 \cdot 4 \cdot 66 \cdot 16 = 54{,}912.$$

(b) In the above procedure, we don't have to divide by two, because we chose the two single cards together ($\binom{12}{2}$). If instead we picked the two single cards separately ($12 \cdot 11$ ways), then we would need to divide by 2, because the single cards' order doesn't matter.

Problem 2.

(a) How many surjective functions are there from [10] to [3]?

(b) How many surjective functions are there from $[n]$ to [3], for $n \geq 1$?

Solution. (a) There are 3^{10} functions $[10] \to [3]$ in total. It's easier to count the non-surjective functions and subtract them from 3^{10}. Let A_1 be the functions that don't map to 1, A_2 those that don't map to 2, etc.

The set of non-surjective functions is $A_1 \cup A_2 \cup A_3$. To find the size of this set, we use PIE.

- What is $|A_i|$? A_1 maps all inputs in $[10]$ to $\{2, 3\}$, so choosing a function in A_1 is a 10-step process with 2 choices at each step: 2^{10} ways. So $|A_i| = 2^{10}$ for all i.
- What is $|A_i \cap A_j|$? For example, $A_1 \cap A_2$ cannot map to 1 or 2, so all the inputs have to map to 3. There is only 1 way to do this: $|A_i \cap A_j| = 1$.
- What is $|A_1 \cap A_2 \cap A_3|$? There are no functions $[10] \to [3]$ that map to *none* of 1,2, or 3, so this is the empty set, with size 0.

So by inclusion-exclusion,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 3 \cdot 2^{10} - 3 \cdot 1 + 0 = 3069,$$

and the number of surjections is $3^{10} - 3 \cdot 2^{10} + 3 = 55{,}980$.

(b) The same argument with n in place of 10 gives $3^n - 3 \cdot 2^n + 3$.

Problem 3. I have 15 identical candies to give to 4 trick-or-treaters.

(a) How many ways are there to do this?

(b) How many ways are there if no one may get more than 6 candies?

Solution. (a) By stars and bars, with 15 candies and 3 dividers: $\binom{15+3}{3} = \binom{18}{3} = 816$.

(b) We will use inclusion-exclusion to figure out how many ways there are where at least one kid gets ≥ 7 candies.

Let B_i be the set of ways to give candies such that kid i gets ≥ 7 candies. Giving i seven candies up front and distributing the remaining 8 freely shows

$$|B_i| = \binom{8+3}{3} = \binom{11}{3}.$$

What about $|B_i \cap B_j|$? That is, situations where both kids i and j get at least 7 candies? Then this uses up 14 candies and leaves 1 to distribute, so

$$|B_i \cap B_j| = \binom{1+3}{3} = \binom{4}{3} = 4.$$

What about $|B_i \cap B_j \cap B_k|$? This set is empty, because we'd need 21 candies to have 3 kids have at least 7.

By inclusion-exclusion,

$$|B_1 \cup B_2 \cup B_3 \cup B_4| = \sum_{i=1}^{4} |B_i| - \sum_{i \neq j} |B_i \cap B_j| + 0 = \binom{4}{1} \cdot \binom{11}{3} - \binom{4}{2} \cdot 4.$$

so the number of distributions with nobody over 6 is $\binom{18}{3} - 3\binom{11}{3} + 4\binom{4}{2} = 180$.

Problem 4. Let $n \geq 1$. Prove that $[n]$ has the same number of odd-sized and even-sized subsets.

> *Hint*: don't count directly; instead, construct a bijection from odd-size to even-size subsets.

Solution.

Proof. Let E be the collection of subsets of $[n]$ of even size and O the collection of those of odd size. Define $f : E \to O$ by

$$f(S) = \begin{cases} S \setminus \{1\} & 1 \in S \\ S \cup \{1\} & 1 \notin S. \end{cases}$$

That is, f toggles whether 1 belongs to S. Toggling changes $|S|$ by exactly one, so it does send even-sized subsets to odd-sized ones, and f is indeed a function $E \to O$.

The same rule defines a function $g : O \to E$, and toggling twice restores the original set, so $g(f(S)) = S$ and $f(g(T)) = T$. So f has a two-sided inverse and is a bijection, giving $|E| = |O|$. $\square$