Recitation 4: Countability

Solutions

Recall from lecture: a set A is **countable** iff there is an injection $A \to \mathbb{N}$. For non-empty A, this is the same as there being a surjection $\mathbb{N} \to A$. A is **countably infinite** iff it is countable and infinite, and **uncountable** iff it is infinite and not countable. Two sets have the same **cardinality** iff there is a bijection between them.

We proved that $\mathbb{N} \times \mathbb{N}$ is countable — in fact there is a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, which walks the grid in anti-diagonal order — and that $\{0,1\}^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$ are uncountable.

Problem 1. Countable or uncountable? (No proofs.)

(a) $B^\star$, the set of *finite* binary strings.

(b) $\{0,1\}^{\mathbb{N}}$, the set of *infinite* binary strings.

(c) $\mathbb{Q}$.

(d) The irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$.

(e) The set of *finite* subsets of $\mathbb{N}$.

(f) $\mathcal{P}(\mathbb{N})$.

(g) The set of functions $\{1,2,3\} \to \mathbb{N}$.

(h) The set of functions $\mathbb{N} \to \{0,1\}$.

Solution. Countable: (a), (c), (e), (g). Uncountable: (b), (d), (f), (h).

Note that (b) and (h) are the same set written two ways, and that (a) and (b) use the same alphabet but come out differently — the difference is whether the strings are finite.

Problem 2. Prove that if A is countable and $B \subseteq A$, then B is countable.

Solution. Let $h : A \to \mathbb{N}$ be an injection. Define $h' : B \to \mathbb{N}$ by $h'(b) = h(b)$. If $h'(b) = h'(b')$ then $h(b) = h(b')$, so $b = b'$ since h is injective. So h' is an injection $B \to \mathbb{N}$.

Problem 3. Prove that if A and B are countable, then $A \cup B$ is countable.

Solution. If $A \cup B = \varnothing$ it is countable, so assume otherwise; discarding whichever of A, B is empty, we may assume both are non-empty. Then there are surjections $g_A : \mathbb{N} \to A$ and $g_B : \mathbb{N} \to B$.

Define $h : \mathbb{N} \to A \cup B$ by

$$h(n) = \begin{cases} g_A(n/2) & n \text{ even} \\ g_B((n+1)/2) & n \text{ odd.} \end{cases}$$

This is a surjection: any $a \in A$ is $g_A(k)$ for some k, so $a = h(2k)$; any $b \in B$ is $g_B(k)$, so $b = h(2k-1)$. So $A \cup B$ is countable.

Problem 4. Prove that if $A_1, A_2, A_3, \ldots$ are all countable, then $\bigcup_{i=1}^{\infty} A_i$ is countable.

Solution. Discard any A_i that is empty (if all of them are, the union is $\varnothing$, which is countable). For each remaining i, countability of A_i gives a surjection $g_i : \mathbb{N} \to A_i$. Define

$$h : \mathbb{N} \times \mathbb{N} \to \bigcup_{i=1}^{\infty} A_i \qquad \text{via} \qquad h(i,j) = g_i(j).$$

h is a surjection: if $a \in \bigcup_i A_i$ then $a \in A_i$ for some i, and since g_i is a surjection, $a = g_i(j)$ for some j, so $a = h(i,j)$.

Let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be the bijection from lecture. A composition of surjections is a surjection, so $h \circ f : \mathbb{N} \to \bigcup_i A_i$ is a surjection, and the union is countable.

Problem 5. Prove that if A and B are countable then so is $A \times B$.

Solution. If either is empty then $A \times B = \varnothing$. Otherwise take surjections $g_A : \mathbb{N} \to A$ and $g_B : \mathbb{N} \to B$, and define $k : \mathbb{N} \times \mathbb{N} \to A \times B$ by $k(m,n) = (g_A(m), g_B(n))$. Given $(a,b) \in A \times B$, write $a = g_A(m)$ and $b = g_B(n)$; then $k(m,n) = (a,b)$, so k is a surjection. Composing with the bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ gives a surjection $\mathbb{N} \to A \times B$.

Problem 6. Use the previous problems to prove each of the following.

(a) $B^\star$ is countable.

(b) $\mathbb{Z} \times \mathbb{Z}$ is countable.

(c) $\mathbb{Q}$ is countable.

(d) The set of finite subsets of $\mathbb{N}$ is countable.

Solution. (a) $B^\star = \bigcup_{n=1}^{\infty} B_n$, where B_n is the set of length-n binary strings. Each B_n is finite, hence countable, so by Problem 4 the union is countable.

(b) $\mathbb{Z}$ is countable (we gave a bijection $\mathbb{N} \to \mathbb{Z}$ in lecture), so by Problem 5, $\mathbb{Z} \times \mathbb{Z}$ is countable.

(c) The map $\mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ via $(p,q) \mapsto p/q$ is a surjection, since every rational can be written that way. $\mathbb{Z} \times \mathbb{N}$ is countable by Problem 5, so composing a surjection $\mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ with this map gives a surjection $\mathbb{N} \to \mathbb{Q}$.

(d) For $n \geq 0$ let F_n be the set of subsets of $\mathbb{N}$ of size exactly n. Listing a size-n subset in increasing order gives an injection $F_n \to \mathbb{N}^n$, and $\mathbb{N}^n$ is countable by applying Problem 5 repeatedly. So each F_n is countable by Problem 2, and the set of finite subsets is $\bigcup_{n=0}^{\infty} F_n$, countable by Problem 4.

Problem 7. A real number is *algebraic* if it is a root of some nonzero polynomial with integer coefficients; otherwise it is *transcendental*.

(a) Prove that the set of polynomials with integer coefficients is countable.

(b) Prove that the set of algebraic numbers is countable.

(c) Conclude that transcendental numbers exist.

Solution. (a) For each $d \geq 0$, a polynomial of degree at most d is determined by its $d+1$ coefficients, so there is an injection from the set P_d of such polynomials into $\mathbb{Z}^{d+1}$, which is countable by Problem 5. So each P_d is countable, and the set of all polynomials is $\bigcup_{d=0}^{\infty} P_d$, countable by Problem 4.

(b) A nonzero polynomial of degree d has at most d real roots, so each contributes a finite — hence countable — set of algebraic numbers. The algebraic numbers are the union of these sets, one for each polynomial, and by (a) there are countably many polynomials. So by Problem 4 the algebraic numbers are countable.

(c) Suppose every real number were algebraic. Then $\mathbb{R}$ would be countable by (b), contradicting that $\mathbb{R}$ is uncountable. So some real number is not algebraic.

Note what this argument does *not* do: it exhibits no transcendental number at all. It shows one must exist purely by counting.

Problem 8. Every one-element set $\{x\}$ is countable, and $\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$. Why does Problem 4 not show that $\mathbb{R}$ is countable?

Solution. Problem 4 is about a union of $A_1, A_2, A_3, \ldots$ — that is, *countably many* sets, indexed by $\mathbb{N}$. The union above is indexed by $\mathbb{R}$ itself, which is not countable, so the theorem does not apply.

Problem 9. Suppose we list all the finite binary strings as $s_1, s_2, s_3, \ldots$ and try to run the diagonal argument: build a string whose n-th digit is the opposite of the n-th digit of s_n. What goes wrong? What does this tell you about which sets the diagonal argument works on?

Solution. The string we build has a digit for every $n \in \mathbb{N}$, so it is infinite, and therefore it is not an element of $B^\star$. We have not produced a string missing from the list, so there is no contradiction — which is just as well, since $B^\star$ really is countable (Problem 6(a)).

The diagonal argument needs the object it constructs to lie back inside the set in question. The same issue arises if you try to diagonalize $\mathbb{Q}$: the number you build has no reason to be rational.