Recitation 4: Countability

Recall from lecture: a set A is **countable** iff there is an injection $A \to \mathbb{N}$. For non-empty A, this is the same as there being a surjection $\mathbb{N} \to A$. A is **countably infinite** iff it is countable and infinite, and **uncountable** iff it is infinite and not countable. Two sets have the same **cardinality** iff there is a bijection between them.

We proved that $\mathbb{N} \times \mathbb{N}$ is countable — in fact there is a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, which walks the grid in anti-diagonal order — and that $\{0,1\}^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$ are uncountable.

Problem 1. Countable or uncountable? (No proofs.)

(a) $B^\star$, the set of *finite* binary strings.

(b) $\{0,1\}^{\mathbb{N}}$, the set of *infinite* binary strings.

(c) $\mathbb{Q}$.

(d) The irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$.

(e) The set of *finite* subsets of $\mathbb{N}$.

(f) $\mathcal{P}(\mathbb{N})$.

(g) The set of functions $\{1,2,3\} \to \mathbb{N}$.

(h) The set of functions $\mathbb{N} \to \{0,1\}$.

Problem 2. Prove that if A is countable and $B \subseteq A$, then B is countable.

Problem 3. Prove that if A and B are countable, then $A \cup B$ is countable.

Problem 4. Prove that if $A_1, A_2, A_3, \ldots$ are all countable, then $\bigcup_{i=1}^{\infty} A_i$ is countable.

Problem 5. Prove that if A and B are countable then so is $A \times B$.

Problem 6. Use the previous problems to prove each of the following.

(a) $B^\star$ is countable.

(b) $\mathbb{Z} \times \mathbb{Z}$ is countable.

(c) $\mathbb{Q}$ is countable.

(d) The set of finite subsets of $\mathbb{N}$ is countable.

Problem 7. A real number is *algebraic* if it is a root of some nonzero polynomial with integer coefficients; otherwise it is *transcendental*.

(a) Prove that the set of polynomials with integer coefficients is countable.

(b) Prove that the set of algebraic numbers is countable.

(c) Conclude that transcendental numbers exist.

Problem 8. Every one-element set $\{x\}$ is countable, and $\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$. Why does Problem 4 not show that $\mathbb{R}$ is countable?

Problem 9. Suppose we list all the finite binary strings as $s_1, s_2, s_3, \ldots$ and try to run the diagonal argument: build a string whose n-th digit is the opposite of the n-th digit of s_n. What goes wrong? What does this tell you about which sets the diagonal argument works on?