Recitation 3: Functions

Solutions

Recall that to show $f : A \to B$ is a well-defined function, we must prove three things: **totality** ($f(x)$ is defined for every $x \in A$), **existence** ($f(x)$ is in B), and **uniqueness** (there is only one value of $f(x)$). Make sure to say each one explicitly, even if each part is only one sentence long!

How do you prove each of {injective, surjective, not-injective, not-surjective}?

Injective: assume $f(x) = f(y)$, and derive that $x = y$.

Surjective: for an arbitrary y in the codomain, you must *produce* an x in the domain with $f(x) = y$. Do scratch work to figure out what x should be — then, in the actual proof, declare that x, check it is in the domain, and verify $f(x) = y$. The scratch work by itself is not a proof!

Not injective: exhibit two specific $x \neq y$ with $f(x) = f(y)$.

Not surjective: exhibit a specific y, and *prove* that nothing maps to it. ("I couldn't find one" is not a proof.)

Notation: $\mathbb{R}_{\geq 0}$ means the real numbers greater than or equal to 0.

Problem 1. Is $f : \mathbb{N} \to \mathbb{N}$ a well-defined function, where

$$f(x) = \begin{cases} x + 1 & x \text{ is even} \\ x + 3 & x \text{ is odd?} \end{cases}$$

Solution. Yes.

Totality: every natural number is either even or odd, so at least one case applies to every $x \in \mathbb{N}$.

Existence: if $x \in \mathbb{N}$ then $x + 1$ and $x + 3$ are also in $\mathbb{N}$.

Uniqueness: no number is both even and odd, so at most one case applies; and $x + 1$ and $x + 3$ are each determined by x, since $x = y$ implies $x + 1 = y + 1$ and $x + 3 = y + 3$.

Problem 2. Is $g : \mathcal{P}(\mathbb{N}) \setminus \{\varnothing\} \to \mathbb{N}$ a well-defined function, where $g(S) = \min(S)$, the smallest element of S?

Solution. Yes.

Totality: every *non-empty* set of naturals has a least element. (This is called the **well-ordering principle**, and it is equivalent to induction. Note that we excluded $\varnothing$ from the domain precisely because it has no least element.)

Existence: $\min(S) \in S$, and $S \subseteq \mathbb{N}$, so $\min(S) \in \mathbb{N}$.

Uniqueness: suppose m and m' are both least elements of S. Since m is least and $m' \in S$, we get $m \leq m'$; symmetrically $m' \leq m$. So $m = m'$.

Problem 3. Is $h : \mathcal{P}(\mathbb{N}) \setminus \{\varnothing\} \to \mathbb{N}$ a well-defined function, where $h(S) = \max(S)$?

Solution. No *totality*. For instance $h(\mathbb{N})$ is undefined, since $\mathbb{N}$ has no greatest element. (Compare with Problem 2: min and max behave very differently on $\mathbb{N}$, because a non-empty subset can go to infinity in one direction but not the other.)

Problem 4. Is $j : \mathbb{Q} \to \mathbb{Z}$ a well-defined function, where $j(p/q) = p + q$?

Solution. No *uniqueness*. For example, $\frac{1}{2} = \frac{2}{4}$, but $j(\frac{1}{2}) = 1 + 2 = 3$ and $j(\frac{2}{4}) = 2 + 4 = 6$.

Problem 5. Which of these are injective? Which are surjective?

(a) $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ via $f(x) = x^2$

(c) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ via $f(x) = x^2$

(d) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ via $f(x) = x^2$

Solution.

	injective?	surjective?
(a) $\mathbb{R} \to \mathbb{R}$	no	no
(b) $\mathbb{R} \to \mathbb{R}_{\geq 0}$	no	yes
(c) $\mathbb{R}_{\geq 0} \to \mathbb{R}$	yes	no
(d) $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$	yes	yes

So "injective" and "surjective" are properties of a formula *together with* a domain and codomain.

Problem 6. Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{2\}$ via $f(x) = \dfrac{2x}{x-1}$. Show that f is a bijection.

Solution. First, note that f is well defined: it is total because the denominator $x-1$ is non-zero for every x in the domain, and it satisfies existence because $f(x)$ is never equal to 2, since if $\frac{2x}{x-1} = 2$ then $2x = 2x - 2$, so $0 = -2$, a contradiction.

Proof (injective). Suppose $f(x) = f(z)$ for $x, z \in \mathbb{R} \setminus \{1\}$. Then

$$\frac{2x}{x-1} = \frac{2z}{z-1} \implies 2x(z-1) = 2z(x-1) \implies 2xz - 2x = 2xz - 2z \implies -2x = -2z,$$

so $x = z$. □

Proof (surjective). Let $y \in \mathbb{R}\setminus\{2\}$ be arbitrary. (Scratch work: solving $y = \frac{2x}{x-1}$ gives $y(x-1) = 2x$, so $x(y-2) = y$, so $x = \frac{y}{y-2}$.)

Take $x = \frac{y}{y-2}$. This is defined because $y \neq 2$, and it lies in the domain because $\frac{y}{y-2} = 1$ would give $y = y - 2$, which is impossible. Finally,

$$f\left(\frac{y}{y-2}\right) = \frac{\frac{2y}{y-2}}{\frac{y}{y-2} - 1} = \frac{\frac{2y}{y-2}}{\frac{y-(y-2)}{y-2}} = \frac{2y}{y-2} \cdot \frac{y-2}{2} = y,$$

so y is in the image. Since y was arbitrary, f is surjective. □

Being both injective and surjective, f is a bijection.

Problem 7. Let $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ via $g(S) = S \cup \{1\}$. Is g injective? Is it surjective?

Solution. Neither.

Not injective: $g(\varnothing) = \{1\}$ and $g(\{1\}) = \{1\}$, but $\varnothing \neq \{1\}$.

Not surjective: consider $T = \varnothing$. For any $S \in \mathcal{P}(\mathbb{N})$ we have $1 \in S \cup \{1\} = g(S)$, so $g(S)$ is non-empty, and therefore $g(S) \neq \varnothing$. So nothing maps to T. (The same argument works for any T with $1 \notin T$.)