Recitation 3: Functions

Recall that to show $f : A \to B$ is a well-defined function, we must prove three things: **totality** ($f(x)$ is defined for every $x \in A$), **existence** ($f(x)$ is in B), and **uniqueness** (there is only one value of $f(x)$). Make sure to say each one explicitly, even if each part is only one sentence long!

How do you prove each of {injective, surjective, not-injective, not-surjective}?

Injective: assume $f(x) = f(y)$, and derive that $x = y$.

Surjective: for an arbitrary y in the codomain, you must *produce* an x in the domain with $f(x) = y$. Do scratch work to figure out what x should be — then, in the actual proof, declare that x, check it is in the domain, and verify $f(x) = y$. The scratch work by itself is not a proof!

Not injective: exhibit two specific $x \neq y$ with $f(x) = f(y)$.

Not surjective: exhibit a specific y, and *prove* that nothing maps to it. ("I couldn't find one" is not a proof.)

Notation: $\mathbb{R}_{\geq 0}$ means the real numbers greater than or equal to 0.

Problem 1. Is $f : \mathbb{N} \to \mathbb{N}$ a well-defined function, where

$$f(x) = \begin{cases} x + 1 & x \text{ is even} \\ x + 3 & x \text{ is odd?} \end{cases}$$

Problem 2. Is $g : \mathcal{P}(\mathbb{N}) \setminus \{\varnothing\} \to \mathbb{N}$ a well-defined function, where $g(S) = \min(S)$, the smallest element of S?

Problem 3. Is $h : \mathcal{P}(\mathbb{N}) \setminus \{\varnothing\} \to \mathbb{N}$ a well-defined function, where $h(S) = \max(S)$?

Problem 4. Is $j : \mathbb{Q} \to \mathbb{Z}$ a well-defined function, where $j(p/q) = p + q$?

Problem 5. Which of these are injective? Which are surjective?

(a) $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ via $f(x) = x^2$

(c) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ via $f(x) = x^2$

(d) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ via $f(x) = x^2$

Problem 6. Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{2\}$ via $f(x) = \dfrac{2x}{x-1}$. Show that f is a bijection.

Problem 7. Let $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ via $g(S) = S \cup \{1\}$. Is g injective? Is it surjective?