Recitation 2: Induction

Solutions

A Bad Proof

Claim. For all n, any group of n horses are all the same color.
"Proof." We induct on n.
Base case. $n = 1$: any group of 1 horse is obviously all the same color.
Inductive step. Fix k, and assume that any group of k horses are all the same color (the induction hypothesis). Consider a group of $k + 1$ horses. Remove any one horse; the remaining k horses are all the same color, by IH. Put that horse back, and remove a different one; the remaining k horses are *also* all the same color, by IH. So all $k + 1$ horses are the same color.
What's wrong with this proof?

Solution. The argument breaks at $k = 1$ (going from groups of size 1 to groups of size 2). The two "remaining k horses" subsets are supposed to share a common horse, linking their colors together, but when $k = 1$, they don't.
In the language of lecture, we proved that $P(1)$ is true, and also that $P(2) \implies P(3)$ and $P(3) \implies P(4)$, etc., but without $P(1) \implies P(2)$ the proof fails.

Induction Practice Problems

Proposition. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution.

Proof by induction. Base case: $n = 1$. Then $\sum_{i=1}^{1} i^2 = 1 = \frac{1(2)(3)}{6}$.
Inductive step. Fix $k \in \mathbb{N}$ and assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ (the inductive hypothesis). Then

$$\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2$$

By the induction hypothesis, $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$. So, this is equal to

$$\begin{aligned}
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\big(k(2k+1) + 6(k+1)\big)}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6},
\end{aligned}$$

which is what we want, since $(k+1)+1 = k+2$ and $2(k+1)+1 = 2k+3$.
By induction, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$. $\square$

Proposition. A chocolate bar with n squares requires exactly $n-1$ breaks to separate into individual squares, no matter what order/how you break them.

Solution.

Proof by strong induction. Here a *break* takes one piece and splits it along a groove into two smaller pieces. Note we need *strong* induction: the first break can split the bar into two pieces of any sizes, not just a piece of size $n-1$ and a single square.
Base case: $n = 1$. A single square is already separated, and $0 = 1 - 1$ breaks are needed.
Inductive step. Fix $n \geq 1$ and assume that for every k with $1 \leq k \leq n$, a piece with k squares takes exactly $k-1$ breaks (the inductive hypothesis). Consider a bar with $n+1$ squares. Whatever we do, we make some first break, splitting it into two pieces of sizes a and b, where $a + b = n + 1$ and $a, b \geq 1$. Since $a, b \geq 1$, both $a \leq n$ and $b \leq n$, so the inductive hypothesis applies to each: the two pieces take $a-1$ and $b-1$ further breaks. In total we use

$$1 + (a-1) + (b-1) = a + b - 1 = (n+1) - 1 = n$$

breaks, no matter which first break we chose.
By strong induction, a bar with n squares takes exactly $n-1$ breaks for all $n \geq 1$. $\square$

Proposition. Find a closed form for $\displaystyle\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right)$, and prove it.

Solution.

Proof by induction. Computing the first few values ($\frac{3}{4}, \frac{2}{3}, \frac{5}{8}, \ldots$) suggests the closed form $\displaystyle\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$.
Base case: $n = 2$. Then $\prod_{i=2}^{2}\left(1 - \frac{1}{i^2}\right) = 1 - \frac{1}{4} = \frac{3}{4} = \frac{2+1}{2(2)}$.
Inductive step. Fix $k \geq 2$ and assume $\prod_{i=2}^{k}\left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k}$ (the inductive hypothesis). Then

$$\prod_{i=2}^{k+1}\left(1 - \frac{1}{i^2}\right) = \left(\prod_{i=2}^{k}\left(1 - \frac{1}{i^2}\right)\right)\left(1 - \frac{1}{(k+1)^2}\right)$$

By the induction hypothesis, the first factor is $\frac{k+1}{2k}$. So, this is equal to

$$\begin{aligned} &= \frac{k+1}{2k} \cdot \frac{(k+1)^2 - 1}{(k+1)^2} = \frac{k+1}{2k} \cdot \frac{k(k+2)}{(k+1)^2} \\ &= \frac{k+2}{2(k+1)}, \end{aligned}$$

which is what we want, since $(k+1) + 1 = k + 2$.
By induction, $\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$ for all $n \geq 2$. $\square$

Proposition. The interior angles of an n-sided polygon sum to $180(n-2)$ degrees.

Proposition. Let F_n denote the nth Fibonacci number. Then $F_n \leq 1.7^n$.

Solution.

Proof by strong induction. Since F_{n+1} depends on both F_n and F_{n-1}, we need two base cases.
Base cases. $F_1 = 1 \leq 1.7$, and $F_2 = 1 \leq 2.89 = 1.7^2$.
Inductive step. Fix $n \geq 2$ and assume $F_k \leq 1.7^k$ for every $1 \leq k \leq n$ (the inductive hypothesis). Then

$$F_{n+1} = F_n + F_{n-1} \leq 1.7^n + 1.7^{n-1} = 1.7^{n-1}(1.7+1) = 1.7^{n-1} \cdot 2.7,$$

using the inductive hypothesis on F_n and F_{n-1}. Since $2.7 \leq 2.89 = 1.7^2$, this is at most $1.7^{n-1} \cdot 1.7^2 = 1.7^{n+1}$, as needed.
By strong induction, $F_n \leq 1.7^n$ for all $n \in \mathbb{N}$. $\square$

The only thing we needed about 1.7 is that $1.7^2 \geq 1.7 + 1$. So the same proof shows $F_n \leq c^n$ for any c with $c^2 \geq c + 1$, i.e. any $c \geq \varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden ratio).