# Recitation 1: Proofs

Solutions

## Negating Statements

Practice negating statements. Here are some examples to negate:

- For all students $x$, $x$ is tall or $x$ is short.
- There is a largest integer. How would you write this using quantifiers?

  **Solution.**
  $$\exists n \in \mathbb{Z} : \forall m \in \mathbb{Z},\ m \leq n.$$
  Negating (note this requires applying De Morgan's law twice!):
  $$\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} : m > n.$$

- All integers are either even or odd.
  $$\forall n \in \mathbb{Z},\ \big(\exists k \in \mathbb{Z} : n = 2k\big) \text{ OR } \big(\exists l \in \mathbb{Z} : n = 2l + 1\big).$$

  **Solution.** Negating:
  $$\exists n \in \mathbb{Z},\ \big(\forall k \in \mathbb{Z} : n \neq 2k\big) \text{ AND } \big(\forall l \in \mathbb{Z} : n \neq 2l + 1\big).$$

## Practice Proofs

**(Easy)**

**Proposition.** Let $r \in \mathbb{R}$. If $r$ is irrational, then $r^{1/5}$ is irrational. (by contrapositive)

**Solution.**

*Proof.* We prove the contrapositive: if $r^{1/5}$ is rational, then $r$ is rational. Suppose $r^{1/5} = m/n$ for integers $m, n$ with $n \neq 0$. Then

$$r = \left(r^{1/5}\right)^5 = \left(\frac{m}{n}\right)^5 = \frac{m^5}{n^5},$$

and since $m^5, n^5 \in \mathbb{Z}$ with $n^5 \neq 0$, $r$ is rational. □

**Proposition.** If $n \in \mathbb{N}$, then $n^2$ leaves a remainder of 0 or 1 when divided by 4. (by cases)

**Solution.**

*Proof.* We consider two cases, based on the parity of $n$.
If $n$ is even, then $n = 2k$ for some $k \in \mathbb{Z}$, so $n^2 = 4k^2$, which leaves remainder 0 when divided by 4.
If $n$ is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$, so

$$n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1,$$

which leaves remainder 1 when divided by 4. $\square$

**Proposition.** If $p \mid q$ and $q \mid r$ then $p \mid r$. (direct)

**Solution.**

*Proof.* Since $p \mid q$, there is $a \in \mathbb{Z}$ with $q = pa$. Since $q \mid r$, there is $b \in \mathbb{Z}$ with $r = qb$. Substituting,

$$r = qb = (pa)b = p(ab),$$

and since $ab \in \mathbb{Z}$, this shows $p \mid r$. $\square$

**(Medium)**

**Proposition.** Assume $\sqrt{6}$ is irrational. Show that $\sqrt{2} + \sqrt{3}$ is irrational. (by contradiction)

**Solution.**

*Proof.* Assume, for the sake of contradiction, that $\sqrt{2}+\sqrt{3}$ is rational, so it equals $m/n$ for integers $m, n$. Then

$$(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{2}\sqrt{3} + 3$$

is also rational (it equals $m^2/n^2$). Then $\sqrt{2}\sqrt{3} = \sqrt{6}$ is also rational, but this is a contradiction, since we are given that $\sqrt{6}$ is irrational. $\square$

**Proposition.** Let $n \in \mathbb{N}$. If $n > 2$ and $n$ is prime, then $n$ is odd. (by contradiction, assuming an even prime $> 2$, or by contrapositive: if $n$ is even, then $n \leq 2$ or $n$ is not prime.)

**Solution.**

*Proof.* Suppose, for the sake of contradiction, that $n > 2$, $n$ is prime, and $n$ is even. Since $n$ is even, $n = 2k$ for some $k \in \mathbb{Z}$. Since $n > 2$, we have $k > 1$; and since $k = n/2 < n$, we have $1 < k < n$. But then $k$ is a positive divisor of $n$ other than 1 and $n$, contradicting the fact that $n$ is prime. $\square$

**Proposition.** For $a, b \in \mathbb{Z}$: $a \mid b$ and $b \mid a$ if and only if $a = b$ or $a = -b$.
Remember that an "if and only if" needs to be proven in both directions!

**Solution.**

*Proof ($\Longrightarrow$).* Since $a \mid b$, there is $k \in \mathbb{Z}$ with $b = ak$. Since $b \mid a$, there is $m \in \mathbb{Z}$ with $a = bm$. Substituting,

$$a = bm = (ak)m = a(km),$$

so $a(1 - km) = 0$.
If $a = 0$: then $b = ak = 0$ too, so $a = b$.
If $a \neq 0$: then $1 - km = 0$, i.e. $km = 1$. Since $k, m$ are integers, this forces $k = m = 1$ or $k = m = -1$. In the first case $b = ak = a$; in the second, $b = ak = -a$. □

*Proof ($\Longleftarrow$).* If $a = b$, then $b = a \cdot 1$ and $a = b \cdot 1$, so $a \mid b$ and $b \mid a$.
If $a = -b$, then $b = a \cdot (-1)$, so $a \mid b$; and $a = b \cdot (-1)$, so $b \mid a$. □

**(Hard)**

Using the following identities about 0, 1, and $-1$:

1. For all $x \in \mathbb{R}$, $1x = x$. (This is the definition of 1.)
2. For all $y \in \mathbb{R}$, $0 + y = y$. (This is the definition of 0.)
3. $-1 + 1 = 0$. (This is the definition of $-1$.)

(a) Prove that for all $z$, $z \cdot 0 = 0$.

(b) Prove that $(-1) \cdot (-1) = 1$.

**Solution.**

*Solution to (a).*

$$\begin{aligned} 0x &= (0 + 0)x && \text{(property 2)} \\ &= 0x + 0x && \text{(distributing)} \end{aligned}$$

Then, subtracting $0x$ from both sides, $0 = 0x$. □

*Solution to (b).*

$$\begin{aligned} & 0 = 0 \\ \Longrightarrow\ & (-1) \cdot 0 = 0 && \text{using (a) above} \\ \Longrightarrow\ & (-1)(1 - 1) = 0 && \text{property 3} \\ \Longrightarrow\ & (-1)(1) + (-1)(-1) = 0 && \text{distributing} \\ \Longrightarrow\ & -1 + (-1)(-1) = 0 && \text{property 1} \\ \Longrightarrow\ & (-1)(-1) = 1 && \text{(adding 1 to both sides)} \end{aligned}$$

□