Recitation 0: Logic, truth tables, even/odd

Solutions

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Practice Problems

1. Let P be the set of prime numbers. Write, using logical operators and quantifiers, the statement that "every prime number greater than 2 is odd." Can you write this two different ways?

   Solution. There are many correct answers, for instance:

   - $\forall n \in P, \text{if } n > 2 \text{ then } (\exists k \in \mathbb{Z} : n = 2k+1)$.
   - $\neg\exists(n \in P : n > 2 \text{ and } (\exists k \in \mathbb{Z} : n = 2k))$.
   - $\forall x \in \mathbb{N}, x > 2 \implies \neg((x \in P) \text{ and } (\exists k \in \mathbb{Z} : x = 2k))$.

   You can also use the notation for "divisible/not divisible by 2," e.g. $\forall n \in P,\ n > 2 \implies 2 \nmid n$.

2. Prove the following equivalences using a truth table:

$$\begin{aligned} \neg\neg p &\equiv p \\ p \implies q &\equiv \neg p \vee q \\ p \implies q &\equiv \neg q \implies \neg p \qquad \text{(contrapositive)} \end{aligned}$$

   Solution. In each table, the two columns being compared agree in every row, so the expressions are equivalent.

   $\neg\neg p \equiv p$:

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

   $p \implies q \equiv \neg p \vee q$:

p	q	$p \implies q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

   $p \implies q \equiv \neg q \implies \neg p$:

p	q	$p \implies q$	$\neg q$	$\neg p$	$\neg q \implies \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Prove the following tautologies using a truth table. A tautology is a logical expression that always evaluates to true, no matter what the arguments (p, q) are.

$$\begin{aligned} p \lor \neg p &\equiv \text{true} \\ ((p \implies q) \land p) \implies q &\equiv \text{true} \qquad \text{(modus ponens)} \\ p \land q \implies p &\equiv \text{true} \\ \text{false} \implies (p \land \neg p) &\equiv \text{true} \end{aligned}$$

Solution. In each table, the final column is T in every row, so the expression is a tautology.

$p \lor \neg p$:

p	$\neg p$	$p \lor \neg p$
T	F	T
F	T	T

$((p \implies q) \land p) \implies q$:

p	q	$p \implies q$	$(p \implies q) \land p$	$((p \implies q) \land p) \implies q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$p \land q \implies p$:

p	q	$p \land q$	$p \land q \implies p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

false $\implies (p \land \neg p)$:

p	$\neg p$	$p \land \neg p$	false $\implies (p \land \neg p)$
T	F	F	T
F	T	F	T

Note that the hypothesis is always false, so the implication is true regardless of what $p \land \neg p$ evaluates to (which happens to also always be false).

3. Decide for each of the following which are true, where P and Q are arbitrary predicates.

(a) $\forall x\,(P(x) \wedge Q(x)) \overset{?}{\iff} \forall x\,P(x) \wedge \forall x\,Q(x)$

Solution. True.

(b) $\forall x\,(P(x) \vee Q(x)) \overset{?}{\iff} \forall x\,P(x) \vee \forall x\,Q(x)$

Solution. False: $\Longleftarrow$ is true but $\Longrightarrow$ is not. For example, let x range over the integers, $P(x)$ be "x is even," and $Q(x)$ be "x is odd."

(c) $\exists x\,(P(x) \vee Q(x)) \overset{?}{\iff} \exists x\,P(x) \vee \exists x\,Q(x)$

Solution. True.

(d) $\exists x\,(P(x) \wedge Q(x)) \overset{?}{\iff} \exists x\,P(x) \wedge \exists x\,Q(x)$

Solution. False: $\Longrightarrow$ is true but $\Longleftarrow$ is not. For example, let $P(x)$ be "$x = 7$" and $Q(x)$ be "$x = 2$."

4. When we factor a quadratic, we often do something like

$$(x - a)(x - b) = 0 \implies x = a, b,$$

but then there's sometimes an extraneous solution. Why? What is the fact that we're using here?

Solution. It's really $(x - a)(x - b) = 0 \implies x = a$ OR $x = b$.

Similarly, when "we square root both sides," we often do something like

$$x^2 = b \implies x = \pm\sqrt{b}.$$

What's the exact fact here?

Solution. It's really $x^2 = b \implies x = \sqrt{b}$ OR $x = -\sqrt{b}$.