Lecture 9 (note): proof that $|A \times A| = |A|^2$

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In class, we talked about proving that $|A \times A| = |A|^2$ for a finite set A. This note is meant to explain the proof a bit.

We used these facts:

Proposition 0.1. *If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is also a bijection.*

Definition 0.2. *The **size** of a finite set A is the unique $n \geq 0$ for which there is a bijection $[n] \to A$.*

We want to prove:

Theorem 0.3. *For a finite set A, $|A \times A| = |A|^2$.*

Example 0.4. For example, for the set $A = \{\circ, \triangle, \square\}$, we have $|A| = 3$, and

$$A \times A = \{(\circ, \circ),\ (\circ, \triangle),\ (\circ, \square),\ (\triangle, \circ),\ (\triangle, \triangle),\ (\triangle, \square),\ (\square, \circ),\ (\square, \triangle),\ (\square, \square)\},$$

so $|A \times A| = 9 = 3^2 = |A|^2$.

Proof. Suppose $|A| = n$. We need to prove that $|A \times A| = n^2$.

By definition, $|A| = n$ means that there exists a bijection $f : A \to [n]$. Similarly, by definition, we need to show that there is a bijection $g : A \times A \to [n^2]$.

The idea is to split it into two functions: $h : A \times A \to [n] \times [n]$, and $j : [n] \times [n] \to [n^2]$. Then, we can compose the two functions to make $g(a) = j(h(a))$: First, we convert the two elements of A into two natural numbers $\leq n$ using h, and then we convert the two natural numbers $\leq n$ to one natural number $\leq n^2$ using j.

If we can come up with bijective h and j, then by Prop. 0.1, $g = j \circ h$ will also be bijective.

To come up with a function $h : A \times A \to [n] \times [n]$, we can make use of $f : A \to [n]$. We have a function that maps elements of A to naturals $\leq n$, so we just use it once on each input of h:

$$h(a, b) = (f(a), f(b)).$$

Example 0.5. We can prove that h is a bijection: how would you do this?

This function takes a 2-tuple (an ordered pair (a, b) where $a \in A, b \in A$) and outputs a 2-tuple (an ordered pair (x, y) where $x \in [n], y \in [n]$.)

To come up with a function $j : [n] \times [n] \to [n^2]$, we use the idea of writing out all the elements of $[n] \times [n]$ in a grid. Counting from the top-left and going across rows, the cell (x, y) is the $(x-1)n + y$th cell.

For $n = 3$, the grid looks like this, with the row x and column y labelling which cell we mean:

	$y=1$	$y=2$	$y=3$
$x=1$	1	2	3
$x=2$	4	5	6
$x=3$	7	8	9

In general, row x starts after the $(x-1)$ full rows above it, each of which used up n cells, so it starts at $(x-1)n+1$ and runs to xn:

	$y=1$	$y=2$	$\cdots$	$y=n$
$x=1$	1	2	$\cdots$	n
$x=2$	$n+1$	$n+2$	$\cdots$	$2n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$x=n$	$(n-1)n+1$	$(n-1)n+2$	$\cdots$	n^2

If we want to write $j(x, y) =$ the number in cell (x, y), this function has a nice closed form:

$$j(x, y) = (x-1)n + y.$$

Example 0.6. We can prove that j is a bijection: how would you do this?

Since h and j are both bijections, by Prop. 0.1, $j \circ h$ is also a bijection. Since $j \circ h$ is a bijection from $A \times A$ to $[n^2]$, that means that (by definition of the size of a set), the size of $A \times A$ is n^2, which is $|A|^2$.

□