Lecture 9: finite sets and countability

October 7, 2026

Recall from last time: g is a **left inverse** of f if $g(f(x)) = x$ for all x, and a **right inverse** if $f(g(y)) = y$ for all y. A function is injective (with nonempty domain) exactly when it has a left inverse, and surjective exactly when it has a right inverse; a function with both is a bijection, and the two one-sided inverses coincide.

We'll also use the following two facts:

Proposition 0.1. *If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is also a bijection.*

Proposition 0.2. *For nonempty sets A, B, if there exists a injective $f : A \to B$, then there exists a surjective $g : B \to A$ and vice versa.*

1 Finite and Infinite Sets

We said before that $|A|$ is the size of a set A, if A is finite. But what does it mean for A to be finite? In fact, how can we formalize the idea of the "size" of a set at all?

For example, $\{1, 2, 3, 4\}$ is clearly finite. The interval $[0, 1]$ is finite-ish, in that it has a start and an end, but it has infinitely many elements. And $[0, \infty)$ is infinite-ish both in length and in number of elements.

Intuitively, a set is finite if you can write down its elements and eventually stop: "the first element of A is 1, the second is 4, the third is 6, ..., the tenth is 48."

Definition 1.1. *A set A is **finite** if there is a bijection between A and $[n]$ for some $n \geq 0$. If A is not finite, we say it is **infinite**.*

Definition 1.2. *The **size** of a finite set A is the unique $n \geq 0$ for which there is a bijection $[n] \to A$.*

Example 1.3. For a finite set A, $|A \times A| = |A|^2$.

Theorem 1.4. *Let X and Y be sets.*

- *If Y is finite and there is an injection $f : X \to Y$, then X is finite and $|X| \leq |Y|$.*
- *If X is finite and there is a surjection $f : X \to Y$, then Y is finite and $|Y| \leq |X|$.*
- *If either X or Y is finite and there is a bijection $f : X \to Y$, then both are finite and $|X| = |Y|$.*

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With this, we can finally say precisely what the x's-and-o's picture was getting at: there's an injection from the o's to the x's, so there are definitely at least as many x's as o's.

These theorems are proved by induction on the sizes of X and Y. This is also how we'd prove the principle of inclusion-exclusion from last week: $|A \cup B| = |A| + |B| - |A \cap B|$.

2 Countably Infinite Sets

What about infinite sets? Some of you may have heard that "not all infinities are the same."

We'd like to talk about the "size" of an infinite set too. There's probably the "same number" of evens as odds — but is there the same number of evens as naturals? Is there the same number of elements of $\mathbb{N}$ as of $\mathcal{P}(\mathbb{N})$?

Definition 2.1. *A set A is **countable** iff there is a injection from A to $\mathbb{N}$.*

By Prop. 0.2, this means that a nonempty set A is also countable iff there is a surjection $\mathbb{N} \to A$.

Definition 2.2. *A set A is **countably infinite** iff it is countable and infinite.*

Example 2.3. Since we gave a bijection $\mathbb{Z} \leftrightarrow \mathbb{N}$ at the start of last lecture, $\mathbb{Z}$ is countably infinite.

3 Uncountable Sets

Are there any sets that can't be bijected with $\mathbb{N}$?

Definition 3.1. *If a set is infinite and not countable, we say it is **uncountably infinite**, or just **uncountable**.*

Theorem 3.2. *$\mathcal{P}(\mathbb{N})$ is uncountable.*

Theorem 3.3. *$\mathbb{N} \times \mathbb{N}$ is countable.*

We'll prove these next lecture.