Lecture 8: function inverses

October 5, 2026

Recall from last time: a function is **well defined** if it satisfies totality, existence, and uniqueness. A function $f : A \to B$ is **injective** if $f(x) = f(y)$ implies $x = y$, and **surjective** if for every $y \in B$ there is some $x \in A$ with $f(x) = y$. A function that is both is called a **bijection**.

Example 0.1. Give a bijection from $\mathbb{Z}$ to $\mathbb{N}$, and prove that it is a bijection.

1 Inverses

Consider these two functions:

$$f : \mathbb{R} \to \mathbb{R}_{\geq 0} \text{ via } f(x) = x^2, \qquad d : \mathbb{N} \to \mathbb{N} \text{ via } d(n) = 2n.$$

Example 1.1. Is f injective? Is f surjective? What about d? Does either one have an inverse?

It depends on what we mean by "inverse." Consider $f' : \mathbb{R}_{\geq 0} \to \mathbb{R}$ via $f'(x) = \sqrt{x}$. This is *kind of* an inverse to f: it satisfies

$$\forall y \in \mathbb{R}_{\geq 0},\ f(f'(y)) = y, \qquad \text{but } \textit{not} \qquad \forall x \in \mathbb{R}, f'(f(x)) = x,$$

since $f'(f(-2)) = \sqrt{4} = 2 \neq -2$.

Similarly, we're tempted to say that $d'(n) = n/2$ is an inverse to d. As written this isn't well defined — it fails existence, since $5/2 \notin \mathbb{N}$. Instead we could take $d' : \mathbb{N} \to \mathbb{N}$ via $d'(n) = \lceil n/2 \rceil$, or equivalently

$$d'(n) = \begin{cases} n/2 & n \text{ even} \\ (n+1)/2 & n \text{ odd.} \end{cases}$$

Example 1.2. Could we also take $d'(n) = \begin{cases} n/2 & n \text{ even} \\ 1000000 & n \text{ odd} \end{cases}$?

This d' satisfies $d'(d(n)) = n$, but *not* $d(d'(n)) = n$.

Definition 1.3. *Let $f : X \to Y$.*

- *g is a **left inverse** of f if $g(f(x)) = x$ for all $x \in X$. Also called a **post-inverse**, since it's an inverse when applied* after *f.*
- *g is a **right inverse** of f if $f(g(y)) = y$ for all $y \in Y$. Also called a **pre-inverse**, since it's an inverse when applied* before *f.*

If g is both a left and a right inverse of f, we call it ***the*** *inverse of f.*

So f' above is a right inverse of f, and d' is a left inverse of d.

Proposition 1.4. *If f has a right inverse r and a left inverse l, then they are the same: $l = r$.*

2 Inverses, Injectivity, and Surjectivity

There's a close relationship between one-sided inverses and injectivity/surjectivity:

- f is injective (and its domain is nonempty) if and only if f has a left inverse.
- f is surjective if and only if f has a right inverse.

Let's try to prove both of these.

Proof: Injective $\iff$ left inverse. .

($\implies$) Suppose $f : A \to B$ is injective and A is nonempty, so that for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$. Let a_0 be an arbitrary element of A. Define $g : B \to A$ via

$$g(b) = \begin{cases} a & \text{if } b = f(a) \text{ for some } a \in A \\ a_0 & \text{otherwise.} \end{cases}$$

This is well defined because f is injective: there is at most one such a, so the first case never gives two different answers. And $g(f(x)) = x$ by construction, so g is a left inverse.

($\impliedby$) Suppose f has a left inverse g. If $f(x) = f(y)$, then applying g to both sides gives $g(f(x)) = g(f(y))$, i.e. $x = y$. So f is injective. □

Surjective $\iff$ right inverse. .

($\impliedby$) Suppose $f : A \to B$ has a right inverse g. Fix any $y \in B$. To show f is surjective, we must display an $x \in A$ with $f(x) = y$. Take $x = g(y)$; then $f(g(y)) = y$ since g is a right inverse.

($\implies$) We'd like to say: "define g as follows: for any $y \in B$, let x be one of the elements that maps to y (such an x must exist, since f is surjective), and set $g(y) = x$."

As written this isn't total, since it doesn't specify *which* such x we pick. We want to say that it doesn't matter — we can pick any one, as long as we're consistent. (For example, if $A = \mathbb{N}$, we could always pick the smallest x with $f(x) = y$.) But for an arbitrary set A, we can't necessarily do this. And "picking any one" is exactly "constructing a right inverse," so this ends up being circular. □

It turns out we can't prove this direction without the **axiom of choice**:

Definition 2.1 (Axiom of choice)**.** *Fix a set I, and for each $i \in I$ let X_i be a set. Then there exists a function* choose $: I \to \bigcup_{i \in I} X_i$ *such that* choose$(i) \in X_i$ *for all i.*

We won't get too far into this. For the purposes of this class, we'll take as true that f is surjective if and only if it has a right inverse.

Next time we'll use inverses and bijections to say what it means for a set to be finite, and to compare the sizes of infinite sets.