# Lecture 7: functions

September 30, 2026

Recall the "exists unique" quantifier: $\exists! x \in S, P(x)$ means there exists exactly one such element $x$. That is, there is some $x \in S$ with $P(x)$, and moreover if $x, y \in S$ with $P(x)$ and $P(y)$, then $x = y$.

## 1 Functions

You've probably seen functions in high school: a function is a "machine" that takes in inputs and produces outputs. Usually those were functions taking real numbers to real numbers, like

$$f(x) = x^2, \qquad g(x, y) = x^2 + y^2 - 1.$$

We're going to take a more abstract view here.

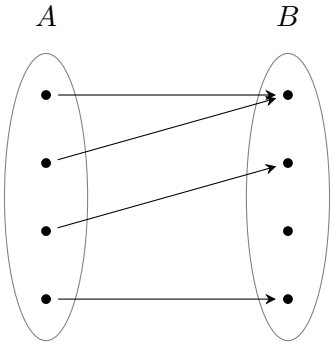
**Definition 1.1.** *For sets $A$ and $B$, a **function** $f$ mapping $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. For $x \in A$, we write $f(x)$ for the element of $B$ assigned to $x$, and we write $f : A \to B$ ("$f$ maps $A$ to $B$").*

To be precise, $f : A \to B$ being a function means that

$$\forall x \in A,\ \exists! y \in B : f(x) = y.$$

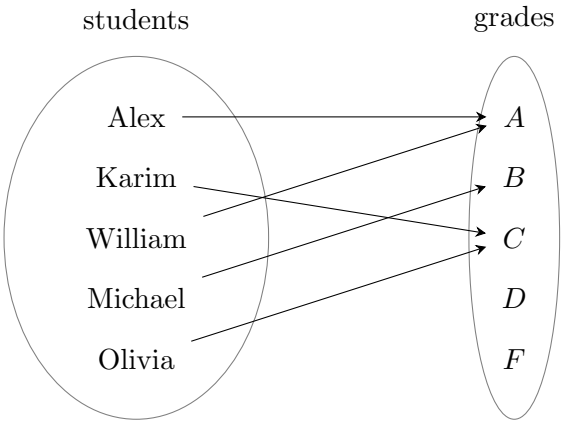
We call $A$ the **domain** and $B$ the **codomain** of $f$.

It's often useful to picture a function as two columns of dots with arrows between them:




The defining property of a function is visible in this picture: *exactly one* arrow leaves each dot of $A$. Nothing is required of the dots of $B$ — some may be hit by several arrows, and some (like the third one here) by none at all.

**Example 1.2.** A function assigning students to grades: $g$ : students $\to \{A, B, C, D, F\}$. The domain is the set of students, and the codomain is the set of possible grades.




## 2 Specifying Functions

Just as sets can be specified in many ways (listing their elements, set-builder notation, and so on), there are many ways to specify a function.

**A list.** The simplest option: just list the output for each input. For example, we could write the students-to-grades function as

$$g(\text{Alex}) = A, \qquad g(\text{Karim}) = C, \qquad \dots$$

**A formula.** When our functions are on numbers, e.g. $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2 + 5$, or other mathematical objects, e.g. $g : \mathbb{N} \to \mathcal{P}(\mathbb{R})$ via $g(n) = [n, n+1)$.

**Cases.** A mix between a list and a formula. For example, $a : \mathbb{R} \to \mathbb{R}$ via

$$a(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

**Example 2.1.** What function is $a$?

**An algorithm or description.** For example, "the area of a polygon" is a function:

$$A : \text{polygons} \to \mathbb{R} \quad \text{via} \quad A(p) = \text{the area of polygon } p.$$

When we specify a function this way, we have to be careful to make sure it is well defined.

*Remark* 2.2. In fact we've been talking about functions all along: $+$ is a function $+ : \mathbb{R}^2 \to \mathbb{R}$, $\wedge$ is a function $\wedge : \{T, F\}^2 \to \{T, F\}$, and the power set is a function

$$\mathcal{P} : \text{sets} \to \text{sets of sets} \quad \text{via} \quad \mathcal{P}(S) = \{X : X \subseteq S\}.$$

Sequences are also implicitly functions with domain $\mathbb{N}$.

## 3 Well-Definedness

In general, we always have to make sure any function we write down is well defined. To be a **well-defined** function, it must have three properties.

**Totality.** $f(x)$ must be defined for all $x$ in the domain.

**Example 3.1.** What goes wrong in each of these?

- $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $f(x, y) = x/y$.

  $f(x, y)$ is not defined when $y = 0$.

- $g :$ students at Columbia $\to \mathbb{N}$ via $g(s) = s$'s grade in Discrete Math.

  $g(s)$ is not defined if $s$ has not taken Discrete Math.

- $h : \mathbb{R} \to \mathbb{R}$ via $h(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$

  $h(0)$ is not defined.

One can also talk about the totality of code: if a program never returns, or throws an exception, then the function it computes isn't total.

**Existence.** $f(x)$ must actually be in the codomain.

**Example 3.2.** $f : \mathbb{N} \to \mathbb{N}$ via $f(n) = \sqrt{n}$ is not a well-defined function because $f(2)$ is not in the codomain, $\mathbb{N}$.

**Uniqueness.** There must be only one value of $f(x)$.

**Example 3.3.** What goes wrong in each of these?

- $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ via $f(S) = S$ without its first element.

  The "first element" is underspecified; we could interpret "first" different ways to get different outputs.

- $g : \mathbb{R} \to \mathbb{R}$ via $g(x)^2 = x$.

- $h : \mathbb{R}^3 \to \mathbb{R}$ via $h(a, b, c) = x$, where $x$ is the solution to $ax^2 + bx + c = 0$.

To prove that a function is well defined, you must prove that it satisfies totality, existence, and uniqueness. We'll do some examples in recitation.

## 4 Proving Functions are Equal

Two functions $f$ and $g$ are equal if and only if

- their domains and codomains match, and
- for all $x$ in the domain, $f(x) = g(x)$.

## 5 Images and Preimages

Suppose everyone in this class passes. Then it's a bit useless for the grades function to have $F$ in its codomain! There's a difference between "grades that it is possible to get" and "grades that are actually achieved."

As another example, let $s : \mathbb{N} \to \mathbb{N}$ via $s(n) = 2n$. Only even numbers are ever output, so why not just say $s : \mathbb{N} \to$ Evens?

Often we want to talk about the values that are *achieved* by a function, but restricting the codomain isn't always convenient — we might have a function where we don't even know which values are achieved. Instead of restricting the codomain, we introduce a word for "the values that are achieved": the **image**.

**Definition 5.1.** *For a function $f : A \to B$ and a subset $U \subseteq A$, the **image** of $U$ under $f$ is*

$$f[U] = \{b \in B : \exists a \in U, f(a) = b\}.$$

*When $U$ isn't specified, the image of $f$ means $f[A]$.*

**Example 5.2.** If all the humanities majors in the class get an $A$ or a $B$, then $g[\text{humanities majors}] = \{A, B\}$. And if everyone passes, then $g[\text{the whole class}] = \{A, B, C, D\}$, so the image of $g$ is $\{A, B, C, D\}$.

There is a matching notion going the other way.

**Definition 5.3.** *For $f : A \to B$ and a subset $V \subseteq B$, the **preimage** of $V$ under $f$ is*

$$f^{-1}[V] = \{a \in A : f(a) \in V\}.$$

**Example 5.4.** $g^{-1}[\{A\}]$ is the set of people who got an $A$.

# 6 Function Composition

In real life the process of grading is many steps long: homeworks and exams and recitation attendance get grades, those are combined and weighted somehow, and *that* is converted into a letter. Written as one function, this would be something like

$$g(x) = \begin{cases} A & 0.4\,(x\text{'s exam grade}) + 0.25\,(x\text{'s hw grade}) + \cdots > 0.9 \\ B & \cdots \end{cases}$$

which is cumbersome, and isn't really how we think about the process. Instead it's a multi-step pipeline:

1. compute $x$'s exam and hw grades,
2. compute the weighted sum of those,
3. convert that to a letter.

Each step is its own function:

$$\begin{aligned} &f_1 : \text{Students} \to \mathbb{R}^2 && \text{via } f_1(x) = (x\text{'s exam score},\ x\text{'s hw score}) \\ &f_2 : \mathbb{R}^2 \to \mathbb{R} && \text{via } f_2(\text{exam}, \text{hw}) = 0.75\,\text{exam} + 0.25\,\text{hw} \\ &f_3 : \mathbb{R} \to \{A, B, C, D, F\} && \text{via } f_3(\text{score}) = \begin{cases} A & \text{score} \geq 0.9 \\ B & 0.8 \leq \text{score} < 0.9 \\ \cdots \end{cases} \end{aligned}$$

So the overall Students $\to \{A, B, C, D, F\}$ function is

$$g(x) = f_3(f_2(f_1(x))).$$

Note the order! We have special notation for "piping functions together" like this, where the output of one directly becomes the input of the next. It's called **function composition**, and we write

$$g = f_3 \circ f_2 \circ f_1.$$

Again, note the order: the function applied *first* is written *last*.

**Example 6.1.** Let $h : A \to B$, $j : C \to D$, and $k : E \to F$. What has to be true of these sets in order to form $h \circ j \circ k$? And what are the domain and codomain of $h \circ j \circ k$?

# 7 Injectivity and Surjectivity

Consider:

xxxxxxxxxxxxx
ooooooooooo

Are there more x's or o's?

Intuitively there are more x's, because we can pair them up: match each o with an x until we run out of o's, and we run out of o's before we run out of x's — so long as each o is assigned a *different* x. A function like this, where each input gets a different output, is called injective.

**Definition 7.1.** *$f : A \to B$ is **injective** iff for all $x, y$, $x \neq y$ implies $f(x) \neq f(y)$. Equivalently (taking the contrapositive), $f(x) = f(y)$ implies $x = y$. An injective function is called an **injection**.*

**Example 7.2.** $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = 2x$ is injective, since $2x = 2y$ implies $x = y$. What's a function from $\mathbb{R}$ to $\mathbb{R}$ that is *not* injective?

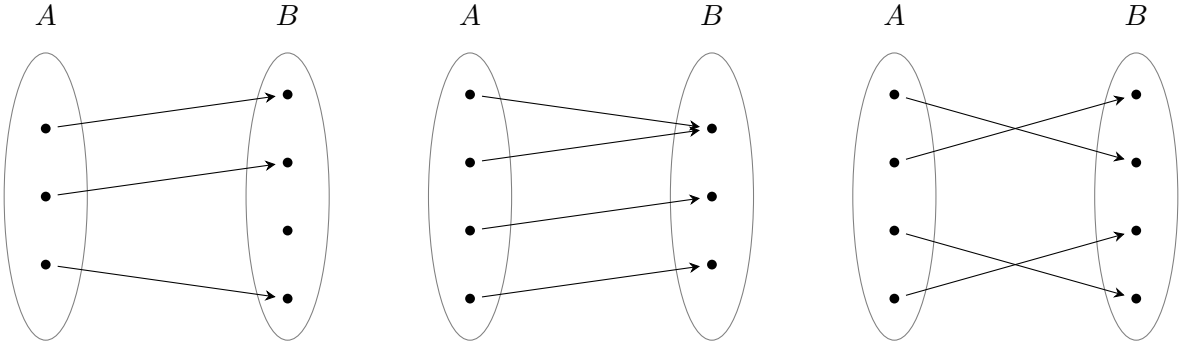
Alternatively, we could have gone the other way and built a function from the x's to the o's. What would have to be true of that function? Every o must be covered by some x. A function like this, where the whole codomain is covered, is called surjective.

**Definition 7.3.** *$g : A \to B$ is **surjective** iff for all $y \in B$, there exists $x \in A$ with $g(x) = y$.*

**Example 7.4.** $g : \mathbb{Z} \to \mathbb{Z}$ via $g(x) = x + 1$ is surjective, since for any $y$ we have $g(y - 1) = y$.

When a function is both injective and surjective, we say it is **bijective**.

In the two-column picture, each of the three properties says something about the arrows arriving in $B$: injective means no dot of $B$ is hit twice, surjective means no dot of $B$ is missed, and bijective means every dot of $B$ is hit exactly once.




injective, not surjective | surjective, not injective | bijective