Lecture 6: set operations and set equality

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Recall from last time: set membership ($\in$), subsets ($\subseteq$) and proper subsets ($\subsetneq$), the empty set $\varnothing$, and the size $|A|$ of a finite set.

1 Two more common types of sets

Intervals. For real numbers a, b, we write $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) for intervals of the real numbers.

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}.$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}.$$

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

A [or] includes the endpoint and a (or) excludes it. For example, $[0, 1]$ is all real numbers between 0 and 1 inclusive, while $[0, 1)$ includes 0 but not 1.

We can also use ∞ or $-\infty$ as an endpoint. Since ∞ isn't itself a real number, that side must always be a (or), e.g. $[1, \infty)$ and never $[1, \infty]$.

Naturals up to n For $n \in \mathbb{N}$, $[n]$ denotes the naturals up to and including n, so $[4] = \{1, 2, 3, 4\}$.

2 Power Sets

Example 2.1. *Let's list out all the subsets of* $\{0, 1, 2\}$*:*

$$\varnothing, \quad \{0\}, \quad \{1\}, \quad \{2\}, \quad \{0, 1\}, \quad \{0, 2\}, \quad \{1, 2\}, \quad \{0, 1, 2\}.$$

Definition 2.2. *The* ***power set*** *of a set A, written $\mathcal{P}(A)$, is the set of all subsets of A.*

Example 2.3. *For a finite set A, what is $|\mathcal{P}(A)|$? And what is $\mathcal{P}(\varnothing)$?*

Note that the empty set is an object, just like any other object, and it's allowed to be a member of a set. So $\{\varnothing\}$ is a one-element set, just like $\{1\}$ is — but $\varnothing$ itself is the empty set, with no elements at all.

3 Set Operations

Definition 3.1. *For sets A and B:*

- *the **union** $A \cup B = \{x : x \in A \text{ or } x \in B\}$;*
- *the **intersection** $A \cap B = \{x : x \in A \text{ and } x \in B\}$.*

Note the similarity to $\vee$ and $\wedge$ — union is an "or," and intersection is an "and."

Example 3.2. *What is $\{1, 2, 3\} \cap \{3, 4, 5\}$? What is $\mathbb{Z} \cap [1, \infty)$?*

We say A and B are **disjoint** if $A \cap B = \varnothing$; that is, if there is no element in both A and B.

Definition 3.3. *The **set difference** and **symmetric difference** are*

$$A \setminus B = \{x : x \in A, x \notin B\}, \qquad A \triangle B = (A \cup B) \setminus (A \cap B).$$

Definition 3.4. *The **complement** of A is $\overline{A} = \{x : x \notin A\}$.*

Taking a complement requires knowing what "everything else" means — that is, it requires specifying a **universal set** U. Usually this is implicit (for instance, "the complement of the evens" probably means all the integers that aren't even, since only integers can be even), but if it's not clear from context, you should always say what U is. In general $\overline{A} = U \setminus A$, so you can usually write something unambiguous like $\mathbb{N} \setminus$ primes instead of $\overline{\text{primes}}$.

Just like for propositions, there are De Morgan's laws for sets:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Finally, the **principle of inclusion-exclusion** counts the size of a union:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

4 Cartesian Products

Definition 4.1. *The **Cartesian product** of A and B is*

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Example 4.2. *Let A be the set of shirts I own, and B the set of pants I own. Then $A \times B$ is the set of (shirt, pant) pairs I can wear — that is, my outfits (assuming I'm a simple dresser who only wears shirts and pants).*

Because it's a "times," we can also take set *powers*: $A^2 = A \times A$ is the set of ordered pairs of elements of A, and similarly for A^k.

Example 4.3. *The (x, y) Cartesian plane is $\mathbb{R}^2$, and the (r, θ) polar coordinate system is $\mathbb{R} \times [0, 2\pi)$. Length-10 lists of natural numbers are $\mathbb{N}^{10}$. In computer hardware, a "byte" is $\{0, 1\}^8$.*

For finite sets, $|A \times B| = |A|\,|B|$.

Example 4.4. *Is $(A \cup B) \times C = (A \times C) \cup (B \times C)$?*

We call the elements of Cartesian products **tuples** (or ordered lists). For instance, $(1, 3, 5)$ is a 3-tuple. Tuples are ordered, have a fixed length, and may contain repeats. For example $(1, 3, 4) \in \mathbb{N}^3$, and it's also in $\mathbb{R}^3$. We usually only talk about k-tuples for $k \geq 2$.

5 Indexed Families of Sets

Just as we can have sequences of numbers (like the Fibonacci numbers), we can have sequences of sets. For example,

$$A_i = [i] \text{ for } i \in \mathbb{N} \quad \text{is the sequence} \quad \{1\},\ \{1,2\},\ \{1,2,3\},\ \{1,2,3,4\},\ \dots$$

and $B_i = [i, i+1)$ is the sequence of length-1 intervals on the positive reals.

When we want to take unions or intersections of many sets at once, we use notation similar to big sums and products:

$$\bigcup_{i=1}^{n} C_i = C_1 \cup C_2 \cup \cdots \cup C_n,$$

and likewise $\bigcap$ for intersections.

Example 5.1. *With A_i and B_i as above:*

$$\bigcup_{i=1}^{n} A_i = [n], \qquad \bigcup_{i=1}^{n} B_i = [1, n+1), \qquad \bigcap_{i=1}^{n} A_i = \{1\}, \qquad \bigcap_{i=1}^{n} B_i = \varnothing.$$

Example 5.2. *Let $D_i = \left(0, \frac{1}{i}\right)$ for $i \in \mathbb{N}$. What is $\bigcap_{i=1}^{\infty} D_i$?*

The indexed Cartesian product is written $\prod_i A_i$.

6 Proving Set Equality

What does it mean for two sets to be equal?

Every element of one is also an element of the other, and vice versa, i.e., they have the same elements. Notice this does *not* mean they have the same definition! We could define the set of odd numbers in many different ways, and all of those definitions give the same set.

So, to prove $A = B$, we can either show

$$A \subseteq B \text{ and } B \subseteq A \qquad \text{(a \textbf{double containment} proof)},$$

or show directly that $\forall x \in U, (x \in A \iff x \in B)$.

Example 6.1. *Let E be the set of even integers and O the set of odd integers, and for sets A, B define $A + B = \{a + b : a \in A, b \in B\}$. Proposition: $E = O + O$.*

Proof. We prove containment in both directions.

($\supseteq$) *First, $O + O \subseteq E$.* Let $x \in O + O$. Then $x = a + b$ for some odd a, b, say $a = 2j + 1$ and $b = 2k + 1$. Then

$$x = (2j + 1) + (2k + 1) = 2(j + k + 1),$$

so x is even, i.e. $x \in E$.

($\subseteq$). *Now, $E \subseteq O + O$.* Let $x \in E$. We need to show that $x = a + b$ for some odd a and b. Take $a = 1$ and $b = x - 1$: then a is odd, and since x is even, $b = x - 1$ is odd. So $x = 1 + (x - 1) \in O + O$.

Since each set contains the other, $E = O + O$. □