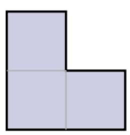
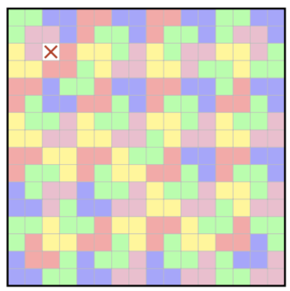
Lecture 5: strong induction, and sets

September 23, 2026

Can you tile a $2^n \times 2^n$ chessboard with one corner removed, using only L-shaped (tromino) pieces?





We can prove this using induction! See the slides for the proof :)

1 Strong Induction

Sometimes, to prove $P(n)$, it's not enough to assume $P(n-1)$: we might need $P(n-2)$, or $P(n/2)$, or even *all* of $P(1), P(2), \ldots, P(n-1)$ at once. This is called **strong induction**.

To prove $\forall n \in \mathbb{N}, P(n)$ by strong induction, it suffices to show

1. one or more **base cases** are true, and
2. for any k, if $P(1), P(2), \ldots, P(k)$ are *all* true, then $P(k+1)$ is also true.

Recall the Fibonacci numbers: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

Proposition 1.1. *For all $n \in \mathbb{N}$, $F_n < 2^{n+1}$.*

Proof. Since the recursive definition of F_n depends on *both* F_{n-1} and F_{n-2}, we'll need two base cases.

Base cases. $F_1 = 1 < 4 = 2^2$, and $F_2 = 1 < 8 = 2^3$.

Inductive step. Fix $n \geq 2$, and suppose $F_k < 2^{k+1}$ for every $1 \leq k \leq n$ (the strong induction hypothesis). Then

$$F_{n+1} = F_n + F_{n-1} < 2^{n+1} + 2^n < 2^{n+1} + 2^{n+1} = 2^{n+2},$$

so $F_{n+1} < 2^{(n+1)+1}$, as needed.

By strong induction, $F_n < 2^{n+1}$ for all $n \in \mathbb{N}$. $\square$

Why do we need *two* base cases here, rather than just one? If we only established $F_1 < 2^2$ and tried to start the inductive step at $n = 1$, we'd need to write $F_2 = F_1 + F_0$, but F_0 was never defined, so the recursive step isn't valid. We need both F_1 and F_2 verified directly before the recursive step (which uses the two *previous* terms) can take over.

Proposition 1.2. *Every natural number ≥ 2 has a prime factorization. That is, for all $n \geq 2$, $n = p_1 \times p_2 \times \cdots p_m$, where $p_1 \cdots p_m$ are all prime (possibly with repeats).*

We can prove this by strong induction as well.

Proof. Base case. $n = 2$ is itself prime, so 2 is (trivially) a prime factorization of itself.

Inductive step. Fix $n \geq 2$, and suppose every k with $2 \leq k \leq n$ has a prime factorization (the strong induction hypothesis). We show $n + 1$ has one too.

Case 1: $n + 1$ is prime. Then $n + 1$ itself is a (one-term) prime factorization of $n + 1$.

Case 2: $n + 1$ is not prime. Then, by definition, $n + 1$ has some positive divisor m other than 1 and $n + 1$ itself, so $2 \leq m \leq n$. Write $n + 1 = mq$ for some integer q. Since $m \geq 2$,

$$q = \frac{n+1}{m} \leq \frac{n+1}{2} \leq n,$$

and since $m \leq n < n + 1$, we also have $q = (n + 1)/m > 1$, so $q \geq 2$. Thus both m and q lie in $\{2, \ldots, n\}$, so by the strong induction hypothesis, each has a prime factorization: $m = p_1 \times \cdots \times p_i$ and $q = p'_1 \times \cdots \times p'_j$. Then

$$n + 1 = mq = p_1 \times \cdots \times p_i \times p'_1 \times \cdots \times p'_j$$

is a prime factorization of $n + 1$.

In either case, $n + 1$ has a prime factorization. By strong induction, every $n \geq 2$ has a prime factorization. $\square$

2 Sets

Definition 2.1. *A **set** is an unordered collection of objects, called its **elements**. Order doesn't matter, and repetition doesn't matter: $\{1, 1, 2, 3\}$ and $\{1, 2, 3\}$ are the same set. We write $x \in A$ to mean "x is an element of A."*

Example 2.2. *Here are some sets:*

$$A = \{1, 2, 3\}; \quad B = \textit{the members of BTS}; \quad C = \{\textit{triangle}, \textit{red}, 5, \textit{Jimin from BTS}\}.$$

The familiar sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{N}$, which we've been using throughout the course, are sets too.

The **empty set**, written $\varnothing$ or $\{\}$, is the set with no elements. Sets can themselves contain other sets — for instance, $\{1, 2, \{1, 2\}, \{1, 2, 3\}\}$ is a (four-element) set whose elements are two numbers and two sets.

For a finite set A, we write $|A|$ for the number of elements of A.

2.1 Set-Builder Notation

Listing out every element works for finite sets, but not for infinite ones. Instead, we use **set-builder notation**:

$$\{x : \text{condition on } x\}.$$

For example, $\{x : x \in \mathbb{Z}, x \geq 0\}$, which we could also write $\{x \in \mathbb{Z} : x \geq 0\}$.

Example 2.3. *How would you write the set of even numbers? How about the set of primes between 100 and 200?*

2.2 Subsets

We already know how to say "7 is in the integers": $7 \in \mathbb{Z}$. But how do we express that the naturals are "contained in" the integers? We can't write $\mathbb{N} \in \mathbb{Z}$, since $\mathbb{N}$ is a set, while $\mathbb{Z}$ contains numbers, not sets. Instead, $\mathbb{N}$ is a **subset** of $\mathbb{Z}$.

Definition 2.4. *A is a **subset** of B, written $A \subseteq B$, if every element of A is also an element of B:*

$$A \subseteq B \iff \forall x, \big(x \in A \implies x \in B\big).$$

Every set is a subset of itself, and the empty set is a subset of every set.

Just as $\leq$ has a strict counterpart $<$, $\subseteq$ has a strict counterpart: A is a **proper subset** of B, written $A \subsetneq B$, if $A \subseteq B$ and there exists some $x \in B$ with $x \notin A$. (We'll avoid the notation $\subset$, since different textbooks use it inconsistently to mean either $\subseteq$ or $\subsetneq$, whereas $\subsetneq$ is unambiguous.)

2.3 Vacuous Statements

A statement of the form "$\forall x \in \varnothing, P(x)$" is always true, **vacuously**. There's no x that could ever serve as a counterexample.

Example 2.5. *"All even primes greater than 5 are divisible by 10" and "all unicorns live in Scotland" are both (vacuously) true.*