Lecture 4: sequences and induction

September 21, 2026

1 Sequences

Here's a different definition of the odd numbers, perhaps closer to what you learned in elementary school: the odd numbers are $1, 3, 5, \ldots$, where the nth odd number is 2 more than the $(n-1)$th odd number. The odd numbers are an example of a **sequence**. One can define a sequence either **directly** (giving a formula a_n in terms of n) or **recursively** (giving a_1, and a formula for a_n in terms of a_{n-1}).

Example 1.1. *The odd natural numbers, defined recursively:* $a_1 = 1$, *and* $a_n = a_{n-1} + 2$ *for* $n > 1$.

Example 1.2. *The Fibonacci numbers, defined recursively:* $a_1 = a_2 = 1$, *and* $a_n = a_{n-1} + a_{n-2}$ *for* $n > 2$.

- For a sequence $a_1, a_2, \ldots, a_n, \ldots$, we write a sum of its terms using the following notation:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

- Similarly, we write a product as follows:

$$\prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times a_{m+2} \times \cdots \times a_n.$$

By convention, an empty sum equals 0 (the additive identity) and an empty product equals 1 (the multiplicative identity).

Example 1.3. *The* ***factorial***, $n! = \prod_{i=1}^{n} i$, *is itself a sequence. Directly:* $a_n = \prod_{i=1}^{n} i$. *Recursively:* $a_1 = 1$, *and* $a_n = n \cdot a_{n-1}$ *for* $n > 1$.

2 Induction

Example 2.1. *What's the sum of the first* n *odd numbers?*

$$1 = 1, \qquad 1 + 3 = 4, \qquad 1 + 3 + 5 = 9, \qquad 1 + 3 + 5 + 7 = 16.$$

These all happen to be perfect squares, which suggests the following.

Conjecture 2.2. *For all* $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} (2i - 1) = n^2.$$

We can prove this using a new technique called **induction**. To prove a statement of the form $\forall n \in \mathbb{N}, P(n)$ (for a predicate P), it suffices to show:

1. $P(1)$ is true (the **base case**), and
2. for any $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is also true (the **inductive step**).

Proof of the Conjecture. Let $P(k)$ be the predicate $\sum_{i=1}^{k}(2i-1) = k^2$.

Base case. $P(1)$ says $\sum_{i=1}^{1}(2i-1) = 2(1)-1 = 1 = 1^2$. ✓

Inductive step. Fix $k \in \mathbb{N}$ and assume $P(k)$ holds — that is, $\sum_{i=1}^{k}(2i-1) = k^2$. (We call this the **inductive hypothesis**.) We want to show $P(k+1)$ holds. Indeed,

$$\sum_{i=1}^{k+1}(2i-1) = \left(\sum_{i=1}^{k}(2i-1)\right) + \big(2(k+1)-1\big) \overset{\text{IH}}{=} k^2 + 2k + 1 = (k+1)^2,$$

so $P(k+1)$ holds.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Why does induction work? When we prove that $\forall n \in \mathbb{N}, P(n)$ using induction, we prove two things: $P(1)$, and for any $k, P(k) \implies P(k+1)$. Together, these let us prove that $P(100)$ is true, or in general $P(n)$ for any n:

$$\big(P(1) \text{ and } (P(1) \implies P(2))\big) \implies P(2), \qquad \big(P(2) \text{ and } (P(2) \implies P(3))\big) \implies P(3), \qquad \ldots$$

Each step only uses the *previous, already-established* case to unlock the next one.

Example 2.3. *Prove that if x is odd, then x^n is odd for every $n \geq 1$.*

Proof.

Base case. x^1 is odd because $x^1 = x$, which is odd by assumption.

Inductive step. We want to show that if x^k is odd, then x^{k+1} is also odd. Fix k and assume that x^k is odd (the induction hypothesis).

Note that $x^{k+1} = x^k \times x$. By the induction hypothesis, x^k is odd. Also, we know x is odd. So, x^{k+1} is odd times odd, which is odd; thus, x^{k+1} is odd.

By the principle of mathematical induction, x^n is odd for all $n \in \mathbb{N}$. □

Example 2.4. *Find a closed form for $\sum_{i=0}^{n} 2^i$, and prove it by induction. (Try guessing the formula first!)*

Proof. We claim that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \geq 0$.

Base case. When $n = 0$, $\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^1 - 1$.

Inductive step. Fix $k \geq 0$ and assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ (the induction hypothesis). We want to show $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$.

Indeed,

$$\sum_{i=0}^{k+1} 2^i = \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} \overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1,$$

as needed.

By the principle of mathematical induction, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \geq 0$. $\square$

A comment. If you're not used to the sum notation, the induction steps of these proofs can look very confusing. Here's an alternative way to write the induction step above:

The induction hypothesis is

$$2^0 + 2^1 + \cdots + 2^k = 2^{k+1} - 1.$$

Then, we need

$$2^0 + 2^1 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1.$$

And this is true, because:

$$\underbrace{2^0 + 2^1 + \cdots + 2^k}_{=2^{k+1}-1 \text{ by IH}} + 2^{k+1} \;=\; (2^{k+1} - 1) + 2^{k+1} \;=\; 2 \cdot 2^{k+1} - 1 \;=\; 2^{k+2} - 1.$$