Lecture 17: discrete probability, events

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1 Probability spaces

Definition 1.1. *A **discrete probability space** consists of*

- *a countable set Ω, called the **sample space**, whose elements are called **outcomes**, and*
- *a function $P : \Omega \to [0, 1]$,*

such that

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

An **event** is a subset $A \subseteq \Omega$. We overload the notation P to apply to events as well as outcomes:

$$P(A) = \sum_{\omega \in A} P(\omega).$$

Example 1.2. I roll a 6-sided die, so $\Omega = \{1, 2, 3, 4, 5, 6\}$. If the die is fair then $P(x) = 1/6$ for every $x \in \Omega$.

"I roll an even number" is an event: it is the subset $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\}$. So

$$P(\text{the roll is even}) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{2}.$$

The simplest probability spaces are the **uniform** ones, where $P(\omega) = 1/|\Omega|$ for every $\omega \in \Omega$. In a uniform space the probability of an event is just a ratio of two counts:

$$P(A) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}.$$

Rolling a fair die gives a uniform space, which is why every event above was a multiple of $1/6$.

Example 1.3. I flip a coin 3 times, so $\Omega = \{H, T\}^3$ and $|\Omega| = 8$. The space is uniform: $P(HHH) = 1/8$, $P(HHT) = 1/8$, and so on for all eight outcomes.

(a) What is the probability that the first two flips are both heads?

(b) What is the probability that the first two flips are both tails?

(c) What is the probability that I get an even number of heads?

2 The birthday problem

There are 23 people in a room. What are the odds that two of them have the same birthday?

Since we are in a uniform space, this is a counting problem. An outcome is an assignment of a birthday to each of the 23 people, so

$$\Omega = [365]^{23}, \qquad |\Omega| = 365^{23},$$

and we suppose that all 365 days are equally likely.

Let A be the event that at least two people share a birthday. Counting $|A|$ directly is unpleasant: the pairs that collide could be any of the pairs, and a birthday could be shared by three or more people. Instead count the complement. Let $\overline{A}$ be the event that *no* two people share a birthday. An outcome in $\overline{A}$ assigns birthdays by a sequential procedure:

- choose person 1's birthday, any of the 365 days — 365 ways
- choose person 2's birthday, any day but person 1's — 364 ways
- choose person 3's birthday, any day but the two already used — 363 ways

$\vdots$

choose person 23's birthday, any day but the 22 already used — 343 ways

so by the multiplication principle

$$|\overline{A}| = 365 \cdot 364 \cdots 343 = \frac{365!}{(365-23)!} = \frac{365!}{342!}.$$

Since A and $\overline{A}$ partition Ω, we have $P(A) = 1 - P(\overline{A})$, and so

$$P(\text{two people share a birthday}) = 1 - \frac{365!/342!}{365^{23}} \approx 1 - 0.4927 = 0.5073,$$

or about 50.7%.

3 Unions of events

Proposition 3.1. *Let A and B be events in a probability space Ω.*

- *If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.*
- *In general, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and therefore $P(A \cup B) \leq P(A) + P(B)$.*
- $P(\Omega) = 1$.

These are direct consequences of counting rules ($|A| + |B| = |A \cup B|$ iff A and B are disjoint, and PIE).

Example 3.2. I flip a coin 3 times, as above. What is the probability that the first two flips are both heads *or* both tails?

4 Nonuniform spaces

Plenty of spaces are not uniform.

Example 4.1. I spin a spinner, where 2/3 of the spinner says "lose" and 1/3 says "win." Then

$$\Omega = \{\text{lose}, \text{win}\}, \qquad P(\text{lose}) = \tfrac{2}{3}, \qquad P(\text{win}) = \tfrac{1}{3}.$$

The sample space can also be infinite, and more abstract.

Example 4.2. Let $\Omega = \mathbb{N}$ with $P(\omega) = 1/2^\omega$. Is this a valid probability space?

It is: the probabilities are all in $[0, 1]$, and they sum to 1, since

$$\sum_{\omega \in \mathbb{N}} \frac{1}{2^\omega} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

Example 4.3. Let $\Omega = \mathbb{N}$ with

$$P(\omega) = \begin{cases} 1 & \omega = 42 \\ 0 & \text{otherwise.} \end{cases}$$

Is this a valid probability space?

5 Nontransitive dice

I have four 6-sided dice, whose faces are

$$A = \{4, 4, 4, 4, 0, 0\}, \quad B = \{3, 3, 3, 3, 3, 3\}, \quad C = \{6, 6, 2, 2, 2, 2\}, \quad D = \{5, 5, 5, 1, 1, 1\},$$

and we play a game: we each pick a die, we both roll, and whoever rolls the higher number wins. Which die is best?

Example 5.1. What is $P(A \text{ beats } B)$?

Every face of B is a 3, so A wins exactly when A rolls a 4, which happens on 4 of its 6 faces. So $P(A \text{ beats } B) = 4/6 = 2/3$.

Example 5.2. What is $P(C \text{ beats } D)$?

Now both dice vary, so take the sample space to be the pairs of rolls,

$$\Omega = C \times D, \qquad |\Omega| = 6 \cdot 6 = 36,$$

which is uniform. Let E be the event that C beats D. To count $|E|$, partition on what C rolls.

- *C rolls a 6.* Then C wins no matter what D rolls. C has 2 faces showing 6 and D has 6 faces, so this gives $2 \cdot 6 = 12$ outcomes.
- *C rolls a 2.* Then C wins exactly when D rolls a 1. C has 4 faces showing 2 and D has 3 faces showing 1, so this gives $4 \cdot 3 = 12$ outcomes.

The two cases are disjoint, and together they cover every outcome in which C wins, so

$$P(C \text{ beats } D) = \frac{|E|}{|\Omega|} = \frac{12+12}{36} = \frac{2}{3}.$$

The same computation on the remaining pairs gives

$$P(A \text{ beats } B) = P(B \text{ beats } C) = P(C \text{ beats } D) = P(D \text{ beats } A) = \frac{2}{3},$$

so "beats" runs in a cycle here: each die beats the next one two times out of three, and A, the die that beats B, loses to D, the die that C beats.

Example 5.3. Are these dice "fair," then? *(Hint: what is the probability that C beats A? What is the probability that D beats B?)*