Lecture 16: Eulerian paths, directed and weighted graphs

November 9, 2026

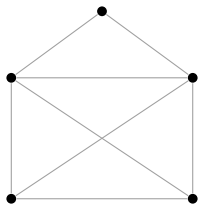
1 Last time

Recall that a **path** is a sequence of vertices in which consecutive ones are adjacent, that it is **simple** if no vertex repeats, and that a **cycle** is a path from a vertex back to itself. We also proved the following, by taking a longest simple path:

Proposition 1.1. *If every vertex of G has degree at least 2, then G contains a simple cycle.*

2 Eulerian Paths and Cycles

Example 2.1. Can you draw this graph without lifting your pencil, and without drawing any edge twice?



Definition 2.2. *A path is **Eulerian** if it uses every edge of the graph exactly once. An **Eulerian cycle** is a cycle that uses every edge exactly once — other sources call this an* Euler circuit*.*

Theorem 2.3. *A connected graph has an Eulerian cycle if and only if every vertex has even degree.*

Proof. ($\implies$) Suppose the graph has an Eulerian cycle. Every time the path passes through a vertex it arrives along one edge and leaves along another, using up two edges at that vertex, and the start and end of the cycle pair up the same way since they are the same vertex. Every edge is used exactly once, so each vertex's edges are used up in pairs, and every degree is even.

($\impliedby$) By strong induction on the number of edges. If there are no edges there is nothing to traverse and the claim holds trivially, so suppose there is at least one edge.

Every degree is even, and every vertex has degree at least 1 by connectedness, so in fact every degree is at least 2. By Prop. 1.1 the graph contains a simple cycle C. Delete the edges of C. Each vertex of the graph loses either 0 or 2 of its edges, so all degrees remain even, and each connected component of what remains has fewer edges than we started with. By the inductive hypothesis, every one of those components has an Eulerian cycle.

Finally, splice them together: walk around C, and whenever you first arrive at a vertex belonging to one of the leftover components, detour around that component's Eulerian cycle — which returns you to where you started — then carry on around C. Every component meets C, since the original graph was connected, so this traversal uses every edge exactly once. □

Theorem 2.4. *A connected graph has an Eulerian path if and only if it has exactly* 0 *or* 2 *vertices of odd degree.*

The 0 case is the theorem above. For the 2 case, the path has to start at one odd-degree vertex and end at the other: those are the only two vertices whose edges can fail to pair up.

Proof. ($\Longleftarrow$) Given a graph G with exactly 2 vertices of odd degree u, v, construct a new graph that adds one vertex w with edges to u and v. Now all vertices have even degree, so a Euler cycle exists. Start and end the cycle at w; the rest of the cycle is an Euler path from u to v.

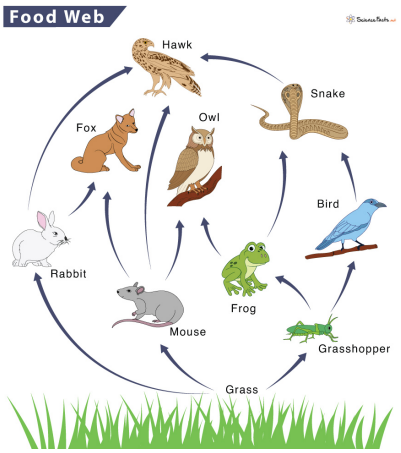
($\Longrightarrow$) Odd-degree nodes can only be visited at the start or end of an Euler path, i.e., there can be at most 2 of them. There can't be one vertex of odd degree due to the handshake lemma. □

Our envelope has exactly two odd-degree vertices — the two bottom corners, of degree 3 — so it can be drawn in one stroke, provided you start at one bottom corner and finish at the other.

Example 2.5. A domino set has $\binom{7}{2}$ dominos, each of which has two sides with numbers from 0 to 6. Can you lay all of them out in a closed ring, so that touching ends match, as in Fig. 1a?



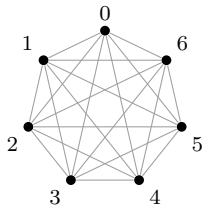
(a) Dominos laid out in a ring, where adjacent numbers match (thank you google images)



(b) A food web

Figure 1: Graphs in the wild. (thank you Google Images)

Let each value be a vertex and each tile be an edge. The dominos are then exactly the edges of K_7, and a ring using every tile once is an Eulerian cycle.

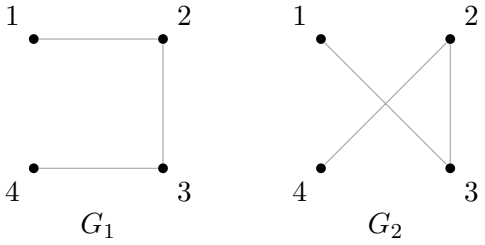


Every vertex of K_7 has degree 6, which is even, so by Theorem 2.3 an Eulerian cycle exists and the ring can be built.

Example 2.6. What if the dominos have numbers from 0 to 5 instead?

3 Isomorphism

Are these two the same graph?



They are *not*: they have the same vertex set but different edge sets. However, there is a sense in which they *are* the same graph: they have the "same shape." Both are a path on four vertices; only the names along the path differ.

Definition 3.1. *Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection $f : V_1 \to V_2$ such that $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$.*

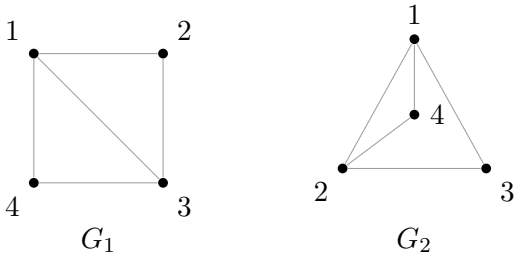
The two graphs above are isomorphic, via

v	1	2	3	4
$f(v)$	1	3	2	4

To check it, push each edge of G_1 through f: $\{1,2\} \mapsto \{1,3\}$, $\{2,3\} \mapsto \{3,2\}$, and $\{3,4\} \mapsto \{2,4\}$. Those are exactly the edges of G_2.

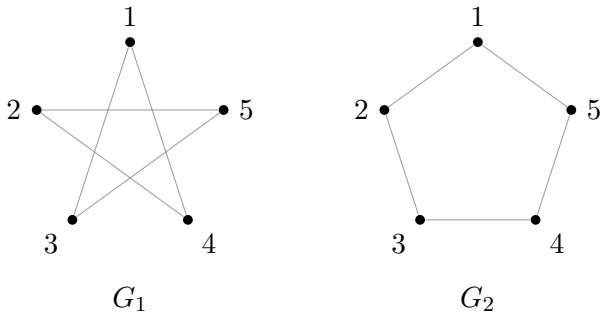
To prove that two graphs *are* isomorphic, we have to exhibit such a bijection. To prove that they are *not*, we find some property that one has and the other lacks.

Example 3.2. Are these isomorphic?



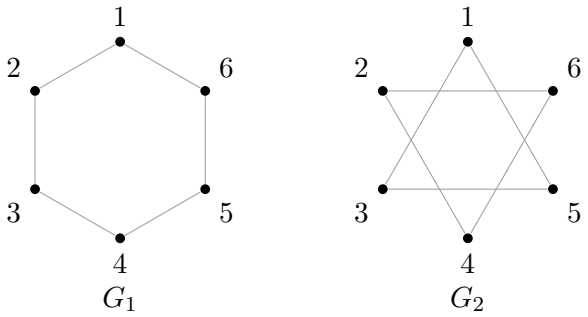
Yes, the bijection is $1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 2$, $4 \mapsto 4$.

Example 3.3. Are these isomorphic?



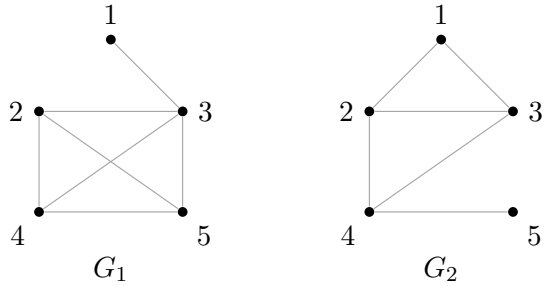
Yes, both are a cycle on five vertices.

Example 3.4. Are these isomorphic?



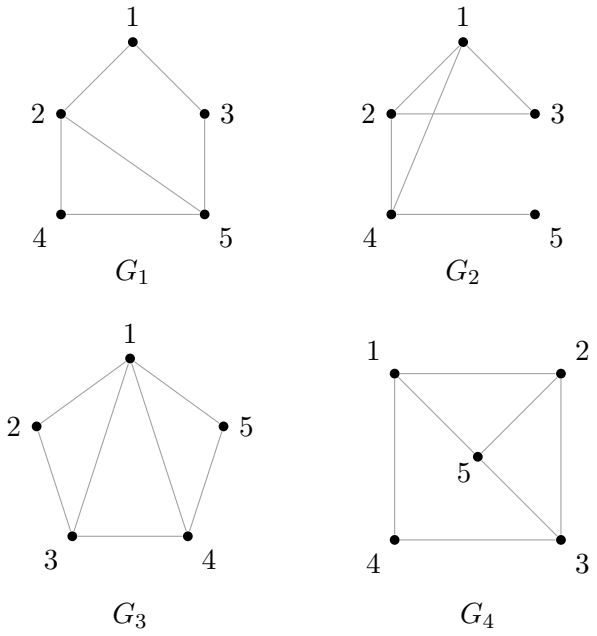
No, G_1 has one connected component and G_2 has two. An isomorphism maps a path between two vertices to a path between their images, so it matches up components one for one.

Example 3.5. Are these isomorphic?



No: G_1 has 7 edges and G_2 has 6. A bijection f matches up the edges of G_1 with the edges of G_2 one for one, so isomorphic graphs have the same number of edges.

Example 3.6. Are G_1 and G_2 isomorphic? They have the same number of vertices and edges. What about G_3 and G_4?

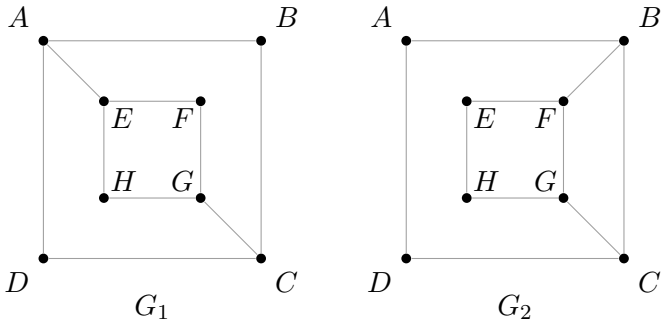


Neither pair are isomorphic. G_1, G_2 both have 5 vertices and 6 edges, but their degrees differ: G_1 has degrees $3, 3, 2, 2, 2$ while G_2 has degrees $3, 3, 3, 2, 1$. In particular G_2 has a vertex of degree 1 and G_1 does not.

G_3, G_4 both have 5 vertices and 7 edges, but G_3 has degrees $4, 3, 3, 2, 2$ and G_4 has degrees $3, 3, 3, 3, 2$. In particular G_1 has a vertex of degree 4 and G_2 does not.

If f is an isomorphism then f matches the neighbors of v with the neighbors of $f(v)$, so $\deg(v) = \deg(f(v))$ for every v. Isomorphic graphs therefore have the same **degree sequence**: the same list of degrees.

Example 3.7. Are these isomorphic?



Here the earlier tests are no help: both graphs have 8 vertices, 10 edges, and degree sequence $3, 3, 3, 3, 2, 2, 2, 2$, but they are still not isomorphic:

- G_1 has 2 simple cycles of length 4, but G_2 has 3.
- Looking at the degree-3 vertices (AEGC in G_1, BFGC in G_2), they induce different subgraphs (two edges in G_1, a cycle in G_2).

Invariants

These things (number of edges, induced subgraphs, degree sequence) are "invariants" that are preserved by isomorphism. Here are some:

If G_1 and G_2 are isomorphic, they must have...	
the same number of vertices	by definition (f is a bijection)
the same number of edges	Example 3.5
the same degree sequence	Example 3.6
the same number of simple cycles of each length	Example 3.7
the same number of connected components	Example 3.4
the same number of cliques of each size	
isomorphic subgraphs induced by the degree-k vertices	Example 3.7

Any one of these that differs is enough to prove two graphs are *not* isomorphic. But no number of them is ever enough to prove that two graphs *are* isomorphic: to prove isomorphism, you have to actually show the bijection.

3.1 Enumerating up to isomorphism

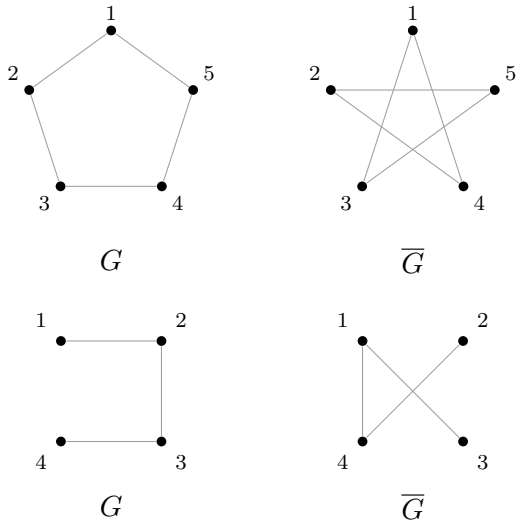
How many distinct graphs (up to isomorphism) are there with 4 vertices? That is, how many graphs are there, where we count two graphs as "the same" if they are isomorphic?

3.2 Graphs isomorphic to their complement

For a graph G, its complement $\overline{G}$ is the graph with the same vertices but the "opposite" edges: $\{u, v\} \in \overline{E} \iff \{u, v\} \notin E$.

Can you draw a graph that is isomorphic to its complement?

Here are two examples:



Proposition 3.8. *No graph with 6 nodes is isomorphic to its complement.*

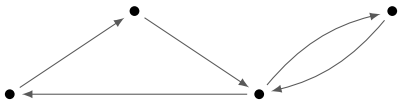
Proof. If there are 6 nodes, there are $\binom{6}{2}$ possible edges. If G has k edges, then $\overline{G}$ has $\binom{6}{2} - k$ edges. In order for G and $\overline{G}$ to be isomorphic, they must have the same number of edges, $\frac{1}{2}\binom{6}{2}$.

But $\binom{6}{2} = 6 \cdot 5/2 = 15$, which is odd, and G can't have $15/2 = 7.5$ edges. □

4 Directed Graphs

So far we've treated edges as bidirectional. But plenty of relationships aren't: "knows about" is one-directional for celebrities, so is "likes," and the airport graph might be as well, if there is a direct flight from a to b but none from b to a.

In a **directed graph** the edges are ordered pairs (a, b) rather than unordered pairs $\{a, b\}$, drawn as arrows. Vertices then have an **in-degree** and an **out-degree** instead of a single degree.



A path in a directed graph has to follow the directions of its edges, and so do simple paths and cycles.

Example 4.1. What is the longest simple path in the food web in Fig. 1b? Does it have any cycles?

A directed graph with no cycles is called *acyclic* (unlike an undirected graph with no simple cycles, which is a tree). Why are they different?

5 Weighted Graphs

Graphs can also be **weighted**: the graph of you and your friends, with each edge weighted by how often you talk, or the graph of the New York subway, with each edge weighted by the ridership between two stations. The edges of a weighted graph are triples (a, b, x) with $x \in \mathbb{R}$.

