Lecture 15: moving around on a graph

paths, cycles; trees; Eulerian paths

November 4, 2026

1 Last time

A graph is a set of vertices together with a set of edges between them. A graph is bipartite if its vertices can be partitioned into two sets A and B such that there are no edges between two vertices in the same set.

A **clique** is a set of vertices all of whose pairwise edges are in the graph, and an **independent set** is a set of vertices none of whose pairwise edges are in the graph.

Proposition 1.1. *Any graph on* 6 *vertices contains either a* 3*-clique or a* 3*-independent set.*

Proof. Consider any vertex v. It has 5 possible edges to the other vertices, and each of those 5 pairs either is an edge or isn't. By the generalized pigeonhole principle (5 pigeons, 2 holes, $5 > 2 \cdot 2$), either at least 3 of them are edges or at least 3 are non-edges.

If v has edges to three vertices A, B, C: if any edge exists among A, B, C — say AB — then v, A, B form a triangle. If not, then A, B, C form an independent set.

If v has non-edges to three vertices A, B, C: if any non-edge exists among A, B, C — say AB — then v, A, B form an independent set. Otherwise A, B, C are pairwise adjacent, and form a triangle. □

2 Paths and Cycles

Definition 2.1. *For a graph $G = (V, E)$, a **path** from u to v is a sequence of vertices*

$$v_0 = u,\ v_1,\ v_2,\ \ldots,\ v_n = v \qquad \text{with } \{v_i, v_{i+1}\} \in E \text{ for all } i.$$

Definition 2.2. *A **simple path** is a path with no repeated vertices.*

Definition 2.3. *A **cycle** is a path from a vertex u back to itself, of length at least* 1*.*

Definition 2.4. *A **simple cycle** is a cycle of length $k \geq 3$ in which $v_0, v_1, \ldots, v_{k-1}$ are all distinct.*

The **length** of a path or cycle is the number of edges in it. So a length-k simple path has $k + 1$ vertices, while a length-k simple cycle has k vertices, since its last vertex is its first one again. A single vertex on its own is a path, and a simple path, of length 0.

3 Connectivity

Definition 3.1. *Two vertices are* ***connected*** *if there is a path between them.*

Proposition 3.2. *Two vertices are connected if and only if there is a* simple path *between them.*

Proof. A simple path is a path, so one direction is immediate. For the other, suppose there is a path from u to v, and among all such paths take a shortest one, $v_0, \ldots, v_n$. If some vertex were repeated, say $v_i = v_j$ with $i < j$, then deleting the steps between them would leave

$$v_0, \ldots, v_i, v_{j+1}, \ldots, v_n,$$

which is still a path from u to v and is shorter — contradicting our choice. So no vertex repeats, and the path is simple. □

Definition 3.3. *A graph is* ***connected*** *if there is a path from u to v for every pair of vertices $u \neq v$.*

Definition 3.4. *For a graph $G = (V, E)$, the* ***connected components*** *are the partition of V in which u and v lie in the same part exactly when u and v are connected.*

A connected graph has exactly one connected component.

Example 3.5. How many connected components does this graph have?



4 Proving Things with Longest Simple Paths

Since our graphs are finite, there are only finitely many simple paths, so among them there is a longest one. This turns out to be useful for proving things about graphs.

Proposition 4.1. *If every vertex of G has degree at least 2, then G contains a simple cycle.*

Proof. Take a longest simple path in G, say $v_1, v_2, \ldots, v_n$. Every neighbor of v_1 must already be on this path: if v_1 had a neighbor w off the path, then $w, v_1, v_2, \ldots, v_n$ would be a longer simple path.

Now, v_1 has degree at least 2, so besides v_2 it has another neighbor v_j on the path, with $j \geq 3$. Then

$$v_1, v_2, \ldots, v_j, v_1$$

is a simple cycle, of length $j \geq 3$. □

5 Trees

Definition 5.1. *A* ***tree*** *is a connected graph with no simple cycles.*

The vertices of a tree come in two kinds: **leaves**, which have degree 1, and **internal vertices**, which have degree greater than 1.

Proposition 5.2. *Every tree with at least 2 vertices has at least two leaves.*

Proof. We could do this by induction, but we'll instead use our new method of using longest simple paths. Take a longest simple path $v_1, \ldots, v_n$ in the tree; since the tree is connected and has at least two vertices, it has an edge, so $n \geq 2$.

Every neighbor of v_1 is on the path, or else we could extend the path and make it longer. But v_1 cannot be adjacent to any v_j with $j \geq 3$: that would give the simple cycle $v_1, v_2, \ldots, v_j, v_1$, and trees have no simple cycles. So v_2 is the only neighbor of v_1, i.e. v_1 is a leaf. The same argument at the other end makes v_n a leaf, and $v_1 \neq v_n$ since the simple path has no repeated vertices. □

Fact 5.3. *If G is a tree and v is a leaf of G, then $G - v$ is also a tree.*

This is what lets us prove things about trees by induction: delete a leaf, apply the induction hypothesis, and put the leaf back.

Proposition 5.4. *Every tree is bipartite.*

Proof. By induction on the number of vertices n.

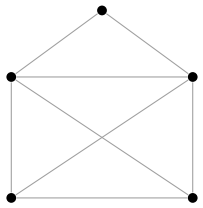
Base case. A tree with $n = 1$ vertex has no edges, so putting its single vertex in A and nothing in B works.

Inductive step. Let T be a tree with $n \geq 2$ vertices, and suppose every tree with $n - 1$ vertices is bipartite. By the proposition above, T has a leaf v; let u be its unique neighbor. By the fact above, $T - v$ is a tree with $n - 1$ vertices, so by the inductive hypothesis its vertices can be split into A and B with no edge inside either part.

Say $u \in A$. Put v into B. The only edge of T not already accounted for is $\{u, v\}$, which now runs between A and B, so no edge lies inside a part. Hence T is bipartite. □

6 Eulerian Paths and Cycles

Example 6.1. Can you draw this graph without lifting your pencil, and without drawing any edge twice?



Definition 6.2. *A path is **Eulerian** if it uses every edge of the graph exactly once. An **Eulerian cycle** is a cycle that uses every edge exactly once — other sources call this an* Euler circuit*.*

Theorem 6.3. *A connected graph has an Eulerian cycle if and only if every vertex has even degree.*

Proof. ($\implies$) Suppose the graph has an Eulerian cycle. Every time the path passes through a vertex it arrives along one edge and leaves along another, using up two edges at that vertex, and the start and end of the cycle pair up the same way since they are the same vertex. Every edge is used exactly once, so each vertex's edges are used up in pairs, and every degree is even.

($\impliedby$) By strong induction on the number of edges. If there are no edges there is nothing to traverse and the claim holds trivially, so suppose there is at least one edge.

Every degree is even, and every vertex has degree at least 1 by connectedness, so in fact every degree is at least 2. By Prop. 4.1 the graph contains a simple cycle C. Delete the edges of C. Each vertex of the graph loses either 0 or 2 of its edges, so all degrees remain even, and each connected component of what remains has fewer edges than we started with. By the inductive hypothesis, every one of those components has an Eulerian cycle.

Finally, splice them together: walk around C, and whenever you first arrive at a vertex belonging to one of the leftover components, detour around that component's Eulerian cycle — which returns you to where you started — then carry on around C. Every component meets C, since the original graph was connected, so this traversal uses every edge exactly once. □

Theorem 6.4. *A connected graph has an Eulerian path if and only if it has exactly* 0 *or* 2 *vertices of odd degree.*

The 0 case is the theorem above. For the 2 case, the path has to start at one odd-degree vertex and end at the other: those are the only two vertices whose edges can fail to pair up.

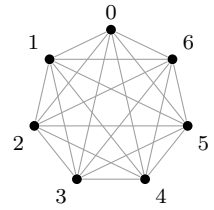
Our envelope has exactly two odd-degree vertices — the two bottom corners, of degree 3 — so it can be drawn in one stroke, provided you start at one bottom corner and finish at the other.

Example 6.5. A domino set has $\binom{7}{2}$ dominos, each of which has two sides with numbers from 0 to 6. Can you lay all of them out in a closed ring, so that touching ends match, as in Fig. 1?



Figure 1: Dominos laid out in a ring, where adjacent numbers match (thank you google images)

Let each value be a vertex and each tile be an edge. The dominos are then exactly the edges of K_7, and a ring using every tile once is an Eulerian cycle.



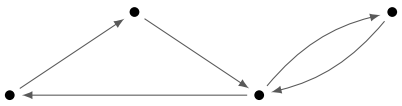
Every vertex of K_7 has degree 6, which is even, so by Theorem 6.3 an Eulerian cycle exists and the ring can be built.

Example 6.6. What if the dominos have numbers from 0 to 5 instead?

7 Directed Graphs

So far we've treated edges as bidirectional. But plenty of relationships aren't: "knows about" is one-directional for celebrities, so is "likes," and the airport graph might be as well, if there is a direct flight from a to b but none from b to a.

In a **directed graph** the edges are ordered pairs (a, b) rather than unordered pairs $\{a, b\}$, drawn as arrows. Vertices then have an **in-degree** and an **out-degree** instead of a single degree.



8 Weighted Graphs

Graphs can also be **weighted**: the graph of you and your friends, with each edge weighted by how often you talk, or the graph of the New York subway, with each edge weighted by the ridership between two stations. The edges of a weighted graph are triples (a, b, x) with $x \in \mathbb{R}$.

