Lecture 14: graphs and subgraphs

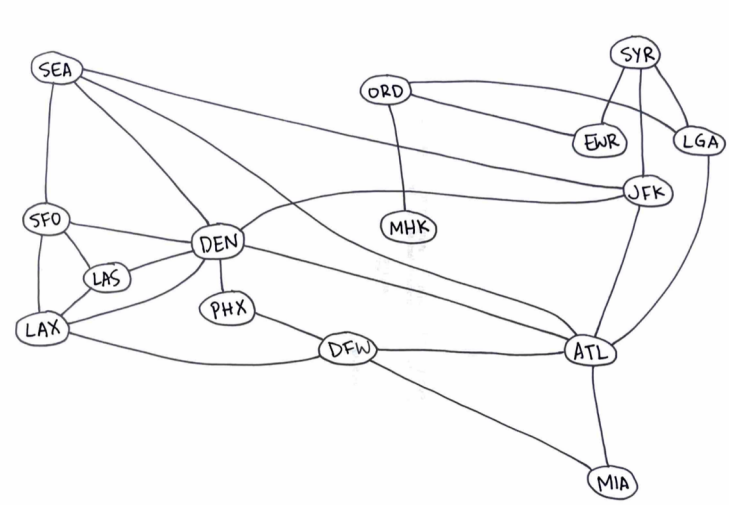
October 28, 2026

Example 0.1. Go around the room and shake hands with however many people you like, keeping track of how many hands you shook. Now raise your hand if you shook an odd number of hands.

Conjecture: an even number of people have their hands raised.

We'll come back to this once we have the right language.

1 Graphs



A graph of airports and flights between them.

A graph consists of **vertices** and **edges** connecting them.

Definition 1.1. *A **graph** is a pair (V, E) — the **vertices** and the **edges** — where E is a set of unordered pairs of vertices.*

V may be finite or infinite; in this class we'll only talk about finite graphs. Written out, the graph above is really

$$\big(\{\text{SFO}, \text{JFK}, \text{LGA}, \text{DEN}, \ldots\},\ \{\{\text{SFO}, \text{DEN}\},\ \{\text{DEN}, \text{JFK}\},\ \ldots\}\big).$$

Definition 1.2. *Two vertices are **neighbors**, or **adjacent**, if there is an edge between them.*

So SFO and DEN are neighbors in the graph above.

Definition 1.3. *The **degree** of a vertex v is the number of edges incident to it, that is, $|\{e \in E : v \in e\}|$.*

For example, SFO has degree 4.

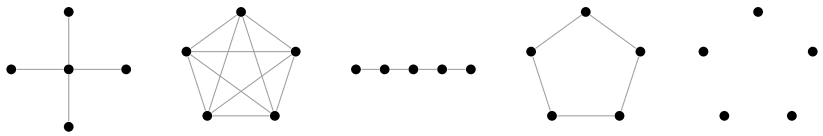
Example 1.4. Which node in the airport graph has the highest degree? Which has the lowest?

Definition 1.5. *A graph is **regular** if all of its vertices have the same degree. If that common degree is d, we say the graph is d-**regular**.*

2 Common Families of Graphs

A few shapes come up often enough to have names.

- the **star graph**: one central vertex joined to all the others;
- the **complete graph** K_n: every pair of vertices is adjacent (e.g., a tight-knit group of friends, or a round-robin tournament);
- the **path graph**: the vertices in a row, each joined to the next (e.g., neighbors along a street, a relay race, a game of telephone);
- the **cycle** C_n: a path whose two ends are joined up (e.g., adjacency among the days of the week, or the months of the year);
- the **empty graph**: no edges at all (e.g., COVID-era in-person interactions).



Example 2.1. How many edges does each of these graphs have, if it has n vertices?

2.1 Specifying a graph

Much as with sets, we can specify a graph either directly — by listing its vertices and edges, or drawing a picture — or by giving a formula.

Example 2.2. The star graph on n vertices is

$$V = \{A, B_1, B_2, \ldots, B_{n-1}\}, \qquad E = \{\{A, B_i\} : i \in [n-1]\}.$$

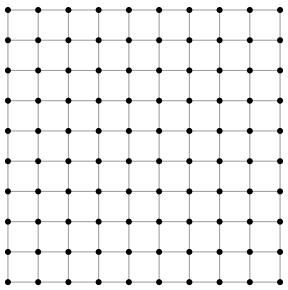
Example 2.3. The airport graph is

$$V = \{\text{SFO}, \ldots\}, \qquad E = \{\{a, b\} : \text{there is a direct flight between } a \text{ and } b\}.$$

Example 2.4. The 10×10 **grid graph** is

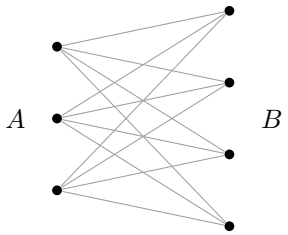
$$V = [10]^2, \qquad E = \big\{\{(x_1, y_1), (x_2, y_2)\} : |x_1 - x_2| + |y_1 - y_2| = 1\big\},$$

that is, two cells have an edge when they differ by 1 in one coordinate.



Definition 2.5. *A graph is **bipartite** if its vertices can be partitioned into two sets A and B such that there are no edges between two vertices in the same set.*

Bipartite graphs show up whenever there are two *types* of vertex: buyers and sellers, or TikTok users and the videos they've watched. The **complete bipartite graph** $K_{m,n}$ has every possible edge between its two sides, with $|A| = m$ and $|B| = n$.



$K_{3,4}$

3 The Handshake Lemma

Theorem 3.1. *In any graph, the sum of the degrees of the vertices is twice the number of edges:*

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof. This is a proof by counting in two ways. The set we'll count is the set of **incidences**: pairs (v, e) where v is a vertex, e is an edge, and v is one of the two endpoints of e.

(LHS) Counting by vertex: each vertex v belongs to exactly $\deg(v)$ edges, by the definition of degree, so grouping the incidences by their vertex gives $\sum_{v \in V} \deg(v)$ of them.

(RHS) Counting by edge: each edge has exactly two endpoints, so grouping the incidences by their edge gives $2|E|$ of them.

So, by counting in two ways, $\sum_{v \in V} \deg(v) = 2|E|$. □

Corollary 3.2. *In any graph, the number of vertices of odd degree is even.*

Proof. Split the sum in the handshake lemma into the even-degree and odd-degree vertices:

$$\underbrace{\sum_{\deg(v) \text{ even}} \deg(v)}_{\text{even, as a sum of evens}} + \sum_{\deg(v) \text{ odd}} \deg(v) = \underbrace{2|E|}_{\text{even}}.$$

So the second sum is even too. A sum of odd numbers is even exactly when there is an even number of them, so there are an even number of odd-degree vertices. □

That settles the question we opened with: the people in the room are the vertices, a handshake is an edge, and the number of hands you shook is your degree. So the number of people who shook an odd number of hands has to be even.

4 Degrees and the Pigeonhole Principle

Example 4.1. Can you draw a graph in which every vertex has a different degree?

Proposition 4.2. *Every finite graph (with at least 2 vertices) where every vertex has at least one neighbor has two vertices of equal degree.*

Proof. Let the graph have $n \geq 2$ vertices. Since all vertices have at least one neighbor, then all n vertices have degree between 1 and $n-1$. By PHP (n pigeons $= n$ vertices, $n-1$ holes $= n-1$ possible degrees), two vertices must have the same degree. □

Remark. The statement is also true if we allow degree-0 ("isolated") vertices. Why?

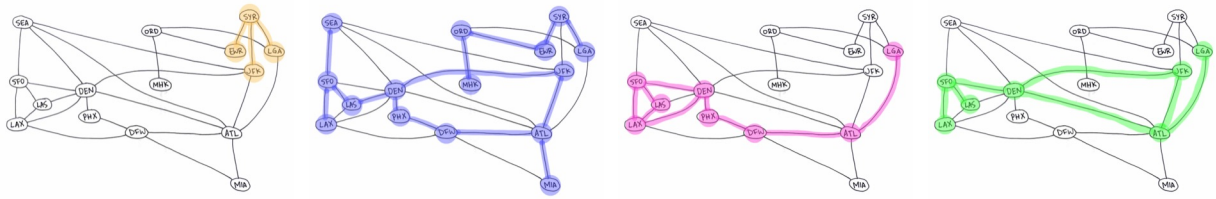
5 Subgraphs

Definition 5.1. *A **subgraph** of a graph $G = (V, E)$ is a graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$.*

Since (V', E') has to be a graph in its own right, the edges in E' may only run between vertices that are in V'.

Here are some subgraphs of the airport graph:

- G_1: the subgraph on just the New York airports, keeping the flights among them;
- G_2: the subgraph whose edges are the routes flown by United;
- G_3: the subgraph whose edges are the routes flown by Southwest;
- G_4: the subgraph whose edges are the flights Gabriel has taken.



G_1: New York G_2: United G_3: Southwest G_4: Gabriel

Definition 5.2. *A subgraph is **spanning** if it includes all of the vertices, that is, if $V' = V$.*

Example 5.3. Which of G_1, G_2, G_3, or G_4 are spanning?

G_2 is: United doesn't fly every route, but every airport in the graph sees a United flight, so no vertices are missing.

Definition 5.4. *For a graph G and a set of vertices $V' \subseteq V$, the subgraph **induced** on V' is (V', E') where*

$$E' = \{\{v_1, v_2\} \in E : v_1 \in V' \text{ and } v_2 \in V'\}.$$

In other words, take the vertices in V' and keep *every* edge of G that runs between two of them.

Example 5.5. Which of G_1, G_2, G_3, or G_4 are induced? What vertex set is it induced by?

G_1 is, induced by the set of New York airports: we kept all the flights between them, rather than selecting some.

Why is G_3 not an induced subgraph?

Definition 5.6. *For a graph G, a **clique** is a set of vertices whose induced subgraph is a complete graph. A clique of size k is a k-**clique**.*

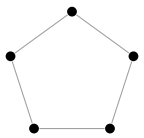
Definition 5.7. *For a graph G, an **independent set** is a set of vertices whose induced subgraph is an empty graph.*

So a clique is a set of vertices that are all adjacent to one another, and an independent set is a set of vertices none of which are adjacent.

Example 5.8. What is the largest clique in the airport graph? What is the largest independent set?

Example 5.9. Try to draw a graph on 5 vertices with no 3-clique and no 3-independent set.

The 5-cycle works:



Example 5.10. Can you draw a graph on 6 vertices with no 3-clique and no 3-independent set?

We'll settle this next time.