Lecture 12: more combinatorics

October 21, 2026

1 Stars and Bars

Example 1.1. I have 10 identical candies, and there are 3 (non-identical) trick-or-treaters at my door. How many ways can I give out the candy? For example, I could give them 3, 3, and 4 pieces, or 10, 0, 0, or 9, 0, 1.

Picture the candy laid out in a row, with a divider marking where one kid's share ends and the next begins. With 10 candies and 3 kids we need 2 dividers, so there are 12 slots in a row, each holding either a candy or a divider:

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This one gives out 3, 6, and 1 candies. Every way of giving out the candy produces exactly one such arrangement, and every arrangement describes exactly one way of giving out the candy, so we can count arrangements instead. An arrangement is fully determined by which 2 of the 12 slots hold the dividers, so there are $\binom{12}{2} = 66$ of them. (Equivalently: the number of anagrams of **********||, similar to last time with "MISSISSIPPI".)

In general, the number of ways to give n identical items to k distinguishable recipients is

$$\binom{n+k-1}{k-1},$$

since n items and $k-1$ dividers fill $n+k-1$ slots, of which we choose the $k-1$ that hold dividers. This is called the **stars and bars** (or balls and walls) argument.

Example 1.2. What if every kid has to get at least one candy? Giving out 10, 0, 0 seems unfair.

Set aside one candy for each kid first, which uses up 3 of them, and then give out the remaining 7 with no constraint. By the previous count there are $\binom{7+2}{2} = \binom{9}{2} = 36$ ways. In general, giving n items to k recipients who each get at least one can be done in

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$$

ways.

Example 1.3. Suppose that $x_1 + x_2 + x_3 + x_4 = 20$, for $x_i \in \mathbb{N}$. How many possible values are there for $x_1 \cdots x_4$?

2 Overcounting

A common strategy is to count a set by describing a process that produces its elements, and then correcting for the fact that the process produces each element more than once. We used this last lecture: there are $\frac{100!}{95!}$ ways to arrange 5 of the 100 pumpkins on the windowsill, and each *choice* of 5 pumpkins arises from 5! of those arrangements.

Similarly, dealing 5 cards from a standard card deck[1] gives

$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311{,}875{,}200$$

sequences, but each 5-card hand is dealt in 5! different orders, so there are $311{,}875{,}200/5! = \binom{52}{5} = 2{,}598{,}960$ hands. So, here we have "overcounted" by 5! times.

Example 2.1. A 5-card hand is *two pair* if it has two cards of one rank, two of another, and a fifth card of a third rank. How many two pair hands are there?

Here is a natural attempt: choose a rank for the first pair (13 ways), choose two suits for it ($\binom{4}{2}$ ways), choose a rank for the second pair (12 ways), choose two suits for it ($\binom{4}{2}$ ways), and choose the fifth card from the 44 cards whose rank we haven't used. That gives

$$13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 44 = 247{,}104.$$

This isn't right! It counts each two pair twice: talking about the "first" and "second" pair puts an order on a pair of ranks, but the order doesn't matter: the hand with two kings and two fives is produced once with the kings chosen first, and again with the fives chosen first. Every hand is produced exactly twice, so we divide by 2!:

$$\binom{13}{2}\binom{4}{2}^2 \cdot 44 = 123{,}552.$$

Example 2.2. How many 5-card hands contain at least one ace?

The naive way to do this is to pick one ace (4 ways) and then 4 other cards (any card is fine, so $\binom{51}{4}$). This overcounts, because hands with more than one ace are counted multiple times. In fact, a hand with 4 aces is counted 4 times!

In this case, dividing out by a constant doesn't work, because some hands are overcounted more than others. Instead, we use the principle of inclusion-exclusion.

3 Inclusion-Exclusion

From our work with sets, we know $|A \cup B| = |A| + |B| - |A \cap B|$. This generalizes to more than two sets.

Example 3.1. What is $|A \cup B \cup C|$?

[1] A standard card deck has 52 cards. Each card has a *suit* and a *rank*. The suits are $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ (called "clubs", "diamonds", "hearts", and "spades"). The ranks are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$. So, a standard card deck is the set *suits* $\times$ *ranks*. Diamonds and hearts are "red suits", and clubs and spades are "black suits". Cards with a rank of J, Q, K, or A are called "face cards."

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

For n sets the pattern is the same: alternately add and subtract the k-way intersections. This is the **principle of inclusion-exclusion**.

Example 3.2. How many 5-card hands contain at least one ace?

Number the aces $1, 2, 3, 4$ and let A_i be the set of hands containing ace i. A hand has at least one ace exactly when it lies in $A_1 \cup A_2 \cup A_3 \cup A_4$. Each intersection is easy to count: fixing j of the aces leaves $5 - j$ cards to choose from the $52 - j$ remaining cards, so

$$|A_i| = \binom{51}{4} = 249{,}900, \qquad |A_i \cap A_j| = \binom{50}{3} = 19{,}600,$$

$$|A_i \cap A_j \cap A_k| = \binom{49}{2} = 1{,}176, \qquad |A_1 \cap A_2 \cap A_3 \cap A_4| = \binom{48}{1} = 48.$$

There are $\binom{4}{k}$ intersections of k sets, so inclusion-exclusion gives

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= \binom{4}{1}\binom{51}{4} - \binom{4}{2}\binom{50}{3} + \binom{4}{3}\binom{49}{2} - \binom{4}{4}\binom{48}{1} \\ &= 999{,}600 - 117{,}600 + 4{,}704 - 48 \\ &= 886{,}656, \end{aligned}$$

which agrees with the count by complement. Notice that the first term is exactly the overcount we got last section: inclusion-exclusion starts from that same flawed count, then subtracts the hands it counted twice, adds back the ones it subtracted too often, and so on.

Example 3.3. How many permutations of [3] map no element to itself? Such a permutation is called a **derangement**.

There are $3! = 6$ permutations in all, so we can count the ones that are not derangements (at least one element maps to itself) and subtract. Let F_i be the set of permutations with $f(i) = i$; then the permutations fixing at least one element are exactly $F_1 \cup F_2 \cup F_3$, and

$$|F_1 \cup F_2 \cup F_3| = |F_1| + |F_2| + |F_3| - |F_1 \cap F_2| - |F_1 \cap F_3| - |F_2 \cap F_3| + |F_1 \cap F_2 \cap F_3|.$$

- $|F_i| = 2$: the value at i is pinned, and the other two elements can be permuted in $2! = 2$ ways.
- $|F_i \cap F_j| = 1$: two of the three values are pinned, so the third one has to be as well.
- $|F_1 \cap F_2 \cap F_3| = 1$: this is the identity permutation.

So $|F_1 \cup F_2 \cup F_3| = 2 + 2 + 2 - 1 - 1 - 1 + 1 = 4$, and there are $3! - 4 = 2$ derangements of [3].

Example 3.4. How many derangements are there of [4]?

4 Bertrand's Ballot Problem

Joseph Bertrand posed this problem in 1887, and proved it by a nasty induction; Désiré André gave a nice combinatorial proof later the same year. We'll use André's nice proof.

Suppose we're counting votes in an election between two candidates, A and B. Candidate A gets a votes and B gets b votes, where $a+b=n$ and $a>b$, so A wins in the end. We count the votes one at a time, keeping a running tally of who's ahead so far. How many orderings of the votes have A in the lead the whole way?

For example, take $n=5$ with $a=3$ and $b=2$. In `AABAB` and `AAABB`, candidate A is always in the lead; in `ABABA` and `BBAAA`, the tally is tied at some point.

Example 4.1. How many orderings of the votes are there in total?

An ordering is determined by which b of the $a+b$ slots hold B's, so there are $\binom{a+b}{b}$.

Every ordering either has a tie at some point or has A in the lead the whole way — note that B cannot be in the lead the whole way, since $b<a$. So it's enough to count the orderings that are tied at some point. André's observation is:

1. Any ordering that starts with a B has a tie at some point.
2. There are as many tied orderings starting with B as there are starting with A.

For (1): if the ordering starts with a B, the tally starts out at 0 to 1 and must end at a to b with $a>b$, so somewhere along the way it has to pass through a tie.

For (2): we pair up the tied orderings starting with A and those starting with B, i.e. we construct a bijection between them. Take a tied ordering and let its *last* tie happen at slot k. Flip every A to a B and every B to an A in the first k slots. Up to slot k the two counts are equal, so flipping leaves the total numbers of A's and B's unchanged, and it swaps which letter the ordering starts with. For example,

$$\underline{\texttt{BBABABA}}\texttt{AAABA} \qquad \longleftrightarrow \qquad \underline{\texttt{AABABABB}}\texttt{AABA}.$$

Flipping twice returns the original, so this is a bijection.

Now, how many orderings start with a B? The first slot is spoken for, so we choose which $b-1$ of the remaining $a+b-1$ slots hold the other B's: there are $\binom{a+b-1}{b-1}$. By (1) all of them are tied, and by (2) there are just as many tied orderings starting with A. So the number of orderings in which A is always in the lead is

$$\binom{a+b}{b} - 2\binom{a+b-1}{b-1}.$$

With $a=3$ and $b=2$ this gives $\binom{5}{2}-2\binom{4}{1}=10-8=2$, which matches the two orderings we found above.

We can also ask what *fraction* of orderings keep A in the lead. Writing the two binomial coefficients out as factorials — and noting that $(a+b-1)-(b-1)=a$, so the same $a!$ appears in both — we get

$$\begin{aligned}
\binom{a+b-1}{b-1} \Big/ \binom{a+b}{b} &= \frac{(a+b-1)!}{(b-1)!\,a!} \cdot \frac{b!\,a!}{(a+b)!} \\
&= \frac{(a+b-1)!}{(a+b)!} \cdot \frac{b!}{(b-1)!} \cdot \frac{a!}{a!} \\
&= \frac{1}{a+b} \cdot b \cdot 1 = \frac{b}{a+b},
\end{aligned}$$

using $(a+b)! = (a+b)\cdot(a+b-1)!$ and $b! = b\cdot(b-1)!$ in the last line. So the fraction is

$$\frac{\binom{a+b}{b} - 2\binom{a+b-1}{b-1}}{\binom{a+b}{b}} = 1 - \frac{2b}{a+b} = \frac{a-b}{a+b}.$$