Lecture 11: counting, permutations, and combinations

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The subfield of math concerned with "counting how many ways there are to do something" is called **combinatorics**. One thing we'll use is the idea of a *partition*:

Definition 0.1. *For a set A, a **partition** of A is a collection of sets B_i where*

$$\bigcup_i B_i = A, \qquad \text{and} \qquad B_i \cap B_j = \varnothing \quad \text{for } i \neq j.$$

So a partition cuts A into pieces that together cover all of A and that don't overlap: every element of A lies in exactly one B_i.

1 Counting and Permutations

A big part of discrete math, and of math in general, is counting how many ways there are to do something. How many paths are there from A to B on a map? How many ways can we assign students to recitations that hold at most 25? How many possible secret Santa assignments are there among n friends?[1]

All of these are really questions about the *sizes of finite sets*.

Example 1.1. I have 100 pumpkins, and I want to arrange them left to right on my porch. How many ways can I do this?

There are 100 choices for the leftmost pumpkin, then 99 for the next, then 98, and so on, so the answer is $100 \cdot 99 \cdots 2 \cdot 1 = 100!$.

Definition 1.2. *A **permutation** of a set A is an ordering of A in which each element appears exactly once.*

For example, the set of permutations of [3] is

$$\{(1,2,3), \quad (1,3,2), \quad (2,1,3), \quad (2,3,1), \quad (3,1,2), \quad (3,2,1)\}.$$

Example 1.3. I have 100 pumpkins, but only a windowsill with space for 5 of them. How many ways are there to arrange pumpkins on it?

There are two ways to count this. We can fill the windowsill left to right: 100 choices for the first spot, 99 for the second, and so on, giving

$$100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 = \frac{100!}{95!}.$$

[1] What if there's a pair of opps can't be assigned to each other at all costs?

Or we can first choose *which* 5 pumpkins to use, and then choose an ordering of those 5, giving (the number of ways to choose 5 out of 100) $\cdot$ 5!. Both count the same set, so if X is the number of ways to choose 5 pumpkins out of 100, then

$$\frac{100!}{95!} = X \cdot 5!, \qquad \text{so} \qquad X = \frac{100!}{95!\,5!}.$$

This number, the number of ways to pick k items out of n possibilities, is called $\binom{n}{k}$.

Definition 1.4. *$\binom{n}{k}$ ("n choose k") is the number of size-k subsets of $[n]$:*

$$\binom{n}{k} := \big|\{B : B \subseteq [n],\ |B| = k\}\big| = \frac{n!}{k!\,(n-k)!}.$$

These are also called the **binomial coefficients**, because they are the coefficients you get when you expand $(x+1)^n$. For example,

$$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1,$$

since $1 = \binom{4}{0}$, $4 = \binom{4}{1}$, $6 = \binom{4}{2}$, $4 = \binom{4}{3}$, and $1 = \binom{4}{4}$.

2 The Multiplication Principle

Both counts above described a *process* for specifying the thing being counted. Choosing an ordering of all 100 pumpkins was a 100-step process:

1. choose the pumpkin that goes leftmost (100 ways);
2. choose the next one (99 ways);
3. choose the next one (98 ways);

$\vdots$

100. choose the last one (1 way);

and the number of ways to do the whole thing was the *product* of the numbers of ways to do each step. Choosing just 5 was a two-step process of the same kind: first choose 5 pumpkins out of the 100 ($\binom{100}{5}$ ways), then choose an ordering of those 5 (5! ways). In general, this is called the **multiplication principle**.

Theorem 2.1 (Multiplication Principle)**.** *Suppose every element of a set A can be uniquely specified by taking a series of steps, where at step i we make a choice among m_i options, and m_i does not depend on the earlier choices. Then $|A| = \prod_i m_i$.*

This is proved by constructing a bijection

$$[\text{choice at step 1}] \times [\text{choice at step 2}] \times \cdots \ \longleftrightarrow\ A,$$

and using the fact that $|A \times B \times C \times \cdots| = |A| \cdot |B| \cdot |C| \cdots$.

Example 2.2. I'm going to carve faces into my pumpkins. I can carve 3 types of eyes, 2 types of nose, and 5 types of mouth, and I can either put a candle inside or not. How many distinct jack-o'-lanterns can I make?

There are $3 \cdot 2 \cdot 5 \cdot 2 = 60$: the set of jack-o'-lanterns is in bijection with

$$\text{eyes} \times \text{noses} \times \text{mouths} \times \{0, 1\},$$

where the last coordinate records whether there's a candle.

3 The Addition Principle

Example 3.1. What if I can carve a face *or* carve a little picture instead? I'm not very artistic, so the only pictures I can manage are numbers of up to two digits. Now how many distinct pumpkins can I carve?

Here we have to do casework. Either the pumpkin gets a face, and there are 60 options, or it gets a number, and there are 100 options, and we add them: $60 + 100 = 160$. In general, this is the **addition principle**.

Theorem 3.2. *If the sets B_i form a partition of A, then $|A| = \sum_i |B_i|$.*

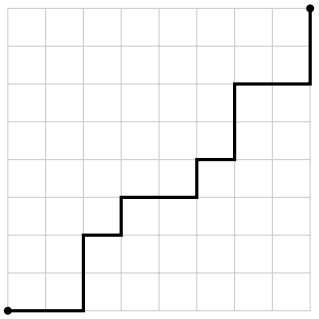
In the example, the set of possible pumpkins is partitioned into two sets, the ones with faces and the ones with numbers, so the size of the whole set is the sum of the two sizes.

Remark 3.3. Problem 2 on HW4 asks you to prove the 2-set version of the addition principle.

4 Paths on a grid

A lot of combinatorics is thinking up clever ways to use these two principles to count interesting sets.

Example 4.1. In an $n \times n$ square city grid, how many paths are there from the bottom-left corner to the top-right corner, if every step has to go up or right?



Reading its steps from the bottom-left corner, the path drawn above is

`R R U U R U R R U R U U R R U U`

Any such path takes exactly n steps right and n steps up (so it has n `R`s and n `U`s), in some order. The path is always $2n$ steps long. So, the path is determined by choosing which n of its $2n$ steps are `R`. There are $\binom{2n}{n}$ ways to do that.

Example 4.2. Suppose n is even. What if the path may not pass through the center intersection of the grid?

