Lecture 1: introduction, propositions, quantifiers

September 9, 2026

Math gives us two things that are especially useful for CS: a way to make precise statements, and a way to show that those statements are true. For example, we might want to know:

- Running this shell command won't destroy my computer.
- Nobody besides me can log in to my email.
- After I run this code, the array will be sorted.

Each of these is vague as stated — what does "destroy" or "sorted" really mean? Math lets us be precise, which lets us more easily argue about whether they are true.

This class has two parts. First, we'll learn the "grammar" of mathematical statements, so that we can speak precisely about objects like graphs, sets, and numbers. Second, we'll learn how to write *proofs*: arguments that establish that a statement is true or false.

1 Propositions

Definition 1.1. *A **proposition** is a statement that is unambiguously either true or false.*

Example 1.2. *The following are propositions:*

- $2 + 4 = 9$ *(false)*
- *All even integers are greater than* 3. *(false)*
- *There are no people in Pupin 301 right now.* *(true or false, depending on the world)*
- *There is at least one odd number.* *(true)*

The following are not *propositions, since they don't have a truth value:*

- 7
- *Solve* $\sqrt{x-3} = x - 5$.
- *This sentence is false.*
- x *is divisible by* 7. *(depends on* x *— see below)*

2 Predicates

Often we want a statement to involve a variable, such as

$$x^2 \geq x, \qquad \text{“}y \text{ is either even or odd,”} \qquad a^2 + b^2 = c^2.$$

Whether these are true depends entirely on what values the variables are allowed to take: in the first, is $x = \frac{1}{2}$? is $x = 2$? A statement like this, with an unspecified (“free”) variable, is called a **predicate** — it is not itself a proposition, since it has no fixed truth value.

Since variables can range over many kinds of objects — numbers, vectors, functions, people, and so on — a predicate must also say what **set** each variable is drawn from.

A **set** is a collection of objects, called its **elements** or **members**. We write $x \in S$ to mean “x is an element of S.” Some common sets that we'll use often:

$$\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\} \quad \text{(integers)}, \qquad \mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{(natural numbers)},$$

$$\mathbb{R} \quad \text{(real numbers)}, \qquad \mathbb{Q} \quad \text{(rational numbers)}.$$

We also use $[n] = \{1, 2, \ldots, n\}$ as shorthand for the natural numbers up to n.

Definition 2.1. *A **predicate** is a statement with one or more free variables, together with a specification of what set each variable is drawn from.*

Example 2.2. *For whole numbers n, let $P(n)$ be the predicate “n is divisible by 7.” Then $P(7)$ is true and $P(8)$ is false — but $P(n)$ itself is neither true nor false until n is specified, so it is not a proposition.*

3 Quantifiers

To turn a predicate into a proposition, we can say something about *all* or *some* of the values its variable could take. This is called **quantification**. There are two main quantifiers:

- $\forall x \in S,\ P(x)$: the **universal quantifier**, read “for all x in S, $P(x)$.”
- $\exists x \in S,\ P(x)$: the **existential quantifier**, read “there exists x in S such that $P(x)$.”

We also use $\exists! x \in S,\ P(x)$ to mean “there exists a *unique* $x \in S$ such that $P(x)$.”

Example 3.1. *Quantifying x differently in $x^2 \geq x$ results in different propositions:*

- *$\forall x \in \mathbb{R}, x^2 \geq x$ is false (take $x = \frac{1}{2}$).*
- *$\forall x \in \mathbb{Z}, x^2 \geq x$ is true.*
- *$\exists x \in \mathbb{R}, x^2 \geq x$ is true.*

Example 3.2. *Quantifiers let us rewrite predicates precisely. For instance,*

$$x \textit{ is even} \quad :\Longleftrightarrow \quad \exists k \in \mathbb{Z}, x = 2k.$$

Example 3.3. *“If $x > 3$ or $y > 4$, then $x + y > 3$” also depends on the quantification. Why?*

4 Logical Connectives

We can combine propositions using **logical operators**, much like $+, -, \times, \div$ combine numbers. Each operator's meaning is captured by a **truth table**.

And, Or, Not

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	$\neg p$
T	F
F	T

Implication The statement $p \implies q$ ("p implies q," or "if p then q") has the truth table

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

The only way for $p \implies q$ to be false is for p to be true and q to be false. In particular, a false hypothesis makes the implication automatically true: "if $2 = 3$, then pigs fly" is (vacuously) true. Implication is one of our most important tools for writing proofs.

Example 4.1. *$\forall x \in \mathbb{R}$, if $x > 10$ then $x > 9$. "If a student is in Pupin 301, then they are on campus."*

If and Only If The statement $p \iff q$ ("p if and only if q") is true exactly when p and q have the same truth value:

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 4.2. *For all real numbers x, $x^2 = 25 \iff (x = 5$ or $x = -5)$.*

5 The Odds-Evens game

Two players, **Odd** and **Even**, simultaneously hold up either 1 or 2 fingers. If the sum is odd, Odd scores that many points; if the sum is even, Even scores that many points. So each round, the possible outcomes are:

- Both play 1: sum is 2 (even), Even scores 2.
- Both play 2: sum is 4 (even), Even scores 4.
- Players differ: sum is 3 (odd), Odd scores 3.

If both players choose randomly (50/50), this seems like a fair game.

Example 5.1. *What do you think would be a good, precise definition of a "fair game"?*

A "theorem" is another word for a proposition, usually used for important (true) results.

Theorem 5.2. *There exists a strategy for Odd such that, against* any *strategy of Even, Odd wins $\frac{1}{12}$ more points per round on average than Even.*

Example 5.3. *How would you define a "strategy"? How would you be precise about this game?*

We'll learn about the building blocks in this proof soon. For now, this is just an example of what a proof can look like:

Proof. Let $p, q \in [0, 1]$. Suppose Odd plays 1 with probability p and 2 with probability $1 - p$, and Even plays 1 with probability q and 2 with probability $1 - q$.

In a given round, Odd wins 3 points with probability $p(1-q)+(1-p)q$, and 0 points otherwise. Even wins 2 points with probability pq, and 4 points with probability $(1 - p)(1 - q)$, (and 0 points otherwise). So Odd's expected point advantage over Even is

$$\begin{aligned}
\text{point advantage for Odd} &= 3p(1 - q) + 3q(1 - p) - 2pq - 4(1 - p)(1 - q) \\
&= 3p - 3pq + 3q - 3pq - 2pq - 4 + 4p + 4q - 4pq \\
&= 7p + 7q - 12pq - 4 \\
&= 7p + q(12p - 7) - 4.
\end{aligned}$$

If Odd chooses $p = \frac{7}{12}$, this becomes

$$7 \cdot \tfrac{7}{12} + q \cdot 0 - 4 = \tfrac{49}{12} - 4 = \tfrac{1}{12},$$

independent of q. Thus Odd has a guaranteed advantage of $\frac{1}{12}$ points per round, no matter the value of q (i.e., what Even does.) $\square$