COMS 3203: Discrete Math, Fall 2026

Problem Set 6: Counting in two ways + graph theory

Due on Friday, November 6, 2026 (different day due to mid-semester break!)

Problem 0 Please put your answers to problem 0 on a different page/image when uploading, so that we don't see your name when grading.

(a) Who did you work with on this problem set?

(b) How long did you spend on this problem set?

Problem 1 Using counting in two ways, show that

$$\binom{n}{a}\binom{n-a}{b} = \binom{n}{a+b}\binom{a+b}{a}$$

Problem 2 Using counting in two ways, show that

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

Problem 3 Everyone in Em and Ori's 10-person board game club is between 15 and 35 years old (inclusive)[1]. The club wants to play a team-vs-team game of Space Cadets. For fairness reasons, the club wants to make sure the two teams have the same average age.

(a) First, show that the club can form two distinct (but potentially overlapping) teams that have the same number of people and the same average age. *Hint: show it for 6-person teams.*

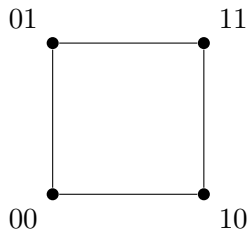
(b) No player can play on both teams at once. Show that the club can form two non-overlapping (disjoint) teams with the same average age.

Problem 4 The *d-dimensional hypercube graph* is defined as $H_d = (V_d, E_d)$, where $V_d = \{0,1\}^d$ and $E_d = \{x, y \in V : x \text{ and } y \text{ differ in exactly one coordinate}\}$.

For example, in the H_4, the vertices 0011 and 1011 are neighbors because they differ by one, but 0000 and 1101 are not neighbors.

This is H_2:

[1] For this problem, assume that all ages are integers.



(a) Draw H_3.

(b) Is H_3 regular? If so, what is its degree? If not, give two nodes with different degrees.

Is H_d regular for all $d \in \mathbb{N}$? If so, what is its degree? If not, give two nodes with different degrees.

(c) Show that H_3 and H_d are bipartite by finding a partition of the vertices V.

(d) A vertex set $U \subseteq V$ is called a *square face* if the subgraph induced by U is the 4-cycle graph. How many square faces are there in H_3? What about in H_d?

Problem 5 Let T be a tree with at least 2 vertices, where no vertices have degree 2 (i.e., all the vertices are either leaves, or have degree ≥ 3). Prove that more than half of the vertices of T are leaves.

Hint: Use the handshake lemma.

Problem 6 (POSTPONED UNTIL HW7) Recall that $K_{m,n}$ (for $m, n \geq 1$) is the complete bipartite graph.

- A path or cycle is *Eulerian* if it uses every edge of the graph exactly once.
- A path or cycle is *Hamiltonian* if it visits every vertex of the graph exactly once (except for the start/endpoint, in the case of the cycle).

(a) Under what conditions (i.e., for what values of m and n) does $K_{m,n}$ have an Eulerian path?

(b) Under what conditions does $K_{m,n}$ have a Eulerian cycle?

(c) Under what conditions does $K_{m,n}$ have a Hamiltonian path?

(d) Under what conditions does $K_{m,n}$ have a Hamiltonian cycle?