COMS 3203: Discrete Math, Fall 2026

Problem Set 3: Sets and Functions

Due on Tuesday October 6, 2026

Problem 0

(a) Who did you work with on this problem set?

(b) How long did you spend on this problem set?

Problem 1 For sets A, X, prove that $X \setminus (X \setminus A) = X \cap A$.

Problem 2 A function $f : A \to A$ is called *idempotent* if $f \circ f = f$. For example, the absolute value function is idempotent, since $||x|| = |x|$ for all x in the reals.[1]

Prove that the only surjective idemoptent function is the identity function $f(x) = x$:

Proposition. *If $f : A \to A$ is idempotent then it is either the identity function $f(x) = x$, or not surjective.*

Problem 3 For $n \in \mathbb{N}$, let B_n be the set of length-n binary strings: $B_n = \{0,1\}^n$. For conciseness, we'll write these as, for example, 1011 instead of $(1, 0, 1, 1)$.

Let

$$B^\star = \bigcup_{i=1}^{\infty} B_i.$$

Which of the following are well-defined functions? If well-defined, prove totality, existence, and uniqueness. If not, explain: what well-definedness property does it lack?

(a) $f_1 : B^\star \to \mathbb{N}$ via $f_1(b) =$ the first position of a 1 in b. For example, $f_1(00110) = 3$.

(b) $f_2 : B^\star \to \mathbb{N}$ via $f_2(b) =$ the length of b. For example, $f_2(00110) = 5$.

(c) $f_3 : \mathbb{N} \to B^\star$ via $f_3(n) = b_0 b_1 b_2 b_3 \cdots b_k$ such that $\sum_{i=0}^{k} 2^i b_i = n$. For example, $f_3(6) = 011$, because $6 = 2 + 4 = 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2$.

(d) $f_4 : B^\star \to \mathbb{Z}$ via $f_4(0) = 0, \quad f_4(1) = 1$, and

$$f_4(b_0 b_1 b_2 \cdots b_k) = \begin{cases} 2 \cdot f_4(b_0 b_1 b_2 \cdots b_{k-1}) + 1 & \text{if } b_k = 0 \\ f_4(b_0 b_1 b_2 \cdots b_{k-1}) & \text{if } b_k = 1 \end{cases} \text{ for } k \geq 1$$

(Hint: use induction.)

Problem 4 Find a bijective function $f : (0,1) \to \mathbb{R}$. Prove that it is well-defined and a bijection.

[1] Idempotent functions are like elevator buttons: pressing many times gets the same result as pressing once.

Problem 5 We say that a set is infinite if it has infinitely many elements. We say that a set is finite if it has finitely many elements.

A subset of the real numbers $S \subseteq \mathbb{R}$ is called *lonely* iff

$$\exists \epsilon \in \mathbb{R} \text{ with } \epsilon > 0 \text{ such that } \forall x \in S, \forall y \in \mathbb{R} \text{ with } y \neq x, (|x - y| < \epsilon \implies y \notin S).$$

(a) Give an example of an infinite set that is lonely. Give an example of a proper subset of the reals that is *not* lonely.

(b) Prove that any finite subset of the reals is lonely.

(c) Find an example of sets $S, T \subseteq \mathbb{R}$ where S and T are both lonely, but $S \cup T$ is not lonely. Prove (for your choice of S and T) that $S \cup T$ is not lonely.[2]

Problem 6 (optional, 1 bonus point) Let S be the (non-empty) set of students in Discrete Math this semester. Prove that there exists a student $\in S$ such that, if that student is wearing a party hat tomorrow, then everyone $\in S$ is wearing a party hat tomorrow.

[2] If you are lonely, take courage from the example of sets S and T!