HW 6 Solutions.

Problem 1. $\binom{n}{a}\binom{n-a}{b} = \binom{n}{a+b}\binom{a+b}{a}$

Proof. By counting in two ways. Both LHS & RHS count the same set: the number of ways to choose, from a set of n balls, a of them to paint red and b of them to paint blue, where no balls are both red & blue.

LHS. A 2-step procedure:

1. Choose a of the balls to paint red. $\binom{n}{a}$ ways.
2. From the remaining $n-a$ balls, choose b to paint blue. $\binom{n-a}{b}$ ways.

By multiplication principle, there are $\binom{n}{a}\binom{n-a}{b}$ ways.

RHS. A 2-step procedure:

1. Choose $a+b$ balls to paint (either color) $\binom{n}{a+b}$ ways
2. From the chosen balls, choose a of them to paint red $\binom{a+b}{a}$ ways

(The other $a+b-a = b$ balls will be painted blue – 1 way)

By mult. princ: there are $\binom{n}{a+b}\binom{a+b}{a}$ ways.

Thus, by C12W, $\binom{n}{a}\binom{n-a}{b} = \binom{n}{a+b}\binom{a+b}{a}$. □.

Problem 2. $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$.

Proof. (By C12W). Both LHS & RHS count the number of ways to add either chocolate or vanilla frosting, or no frosting, to each cupcake in a row of n cupcakes.

RHS: Each cupcake has 3 options (C, V, none) so there are 3^n ways to add flavors.

LHS: We partition on how many cupcakes get frosting. This is a partition b/c it is disjoint (no way to both have k_1 cupcakes w/ frosting and k_2 cupcakes w/ frosting) and covers the set (some # of cupcakes from 0 to n get frosting). Let k be the # of frosted cupcakes, $0 \leq k \leq n$. For any k, there's a 2-step process:

1. Choose which k cupcakes get frosting — $\binom{n}{k}$ ways
2. For each of the k frosted cupcakes, choose Chocolate or vanilla (2 options k times) — 2^k ways

By mult. & add. principles, the total # of ways is $\sum_{k=0}^{n} \binom{n}{k} 2^k$. So, by C12W, LHS = RHS. □.

Problem 3.

(a) We show that there are two size-6 teams with the same total age. (Since the sum of ages are the same, the average is just $\frac{\text{sum}}{6}$, so the averages are also the same).

Since there are 10 members, there are $\binom{10}{6} = 210$ possible teams of 6 people.

Since everyone is at least 15 and at most 35, the smallest possible total age for a team is $15 \times 6 = 90$, and the largest possible total age is $35 \times 6 = 210$.

So, there are at most $210 - 90 + 1 = 121$ possible age-sums but 210 possible teams. By the pigeonhole principle, there must be 2 teams with the same total age, since $210 > 121$. □.

(b). Suppose A and B are two six-person teams w/ the same average age (and thus the same total age).

Consider the teams formed by removing the overlap:

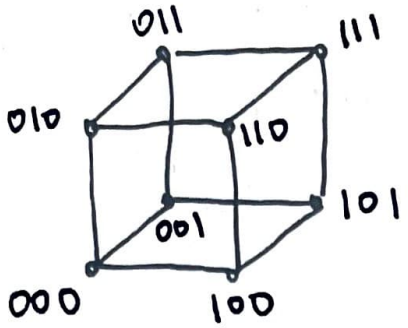
$$A' = A \setminus (A \cap B), \quad B' = B \setminus (A \cap B).$$

A' and B' have the same size $(6 - |A \cap B|)$ and the same total age (total age of A' = total age of A − total age of $A \cap B$ = total age of B − total age of $A \cap B$ = total age of B'), so they have the same average age as well.

Thus, A', and B' are disjoint & have the same avg. age. □.

Problem 4.

(a)



(b) H_3 is regular with degree 3.

For any d, H_d is regular with degree d.

(c). H_3 is bipartite, with partition

$V_1 = \{000, 110, 011, 101\}$

$V_2 = \{001, 100, 010, 111\}$.

H_d is bipartite, with partition

$V_1 = \{X \in \{0,1\}^d : X \text{ has an odd \# of 1's}\}$

$V_2 = \{X \in \{0,1\}^d : X \text{ has an even \# of 1's}\}$

(d) There are 6 square faces in H_3.

In H_d, the square faces are all of the form

$$\left\{ \begin{array}{ccccc} \square & 0 & \square & 0 & \square, \\ \square & 1 & \square & 0 & \square, \\ \square & 0 & \square & 1 & \square, \\ \square & 1 & \square & 1 & \square \end{array} \right\}.$$

where all but two coordinates match.

We can count them via this procedure:

1. Choose 2 indices (among d) to change $\binom{d}{2}$ ways

2. Choose the binary strings for the other $d-1$ positions in the string (2^{d-1} ways)

By the multiplication principle, there are $\binom{d}{2} \cdot 2^{d-1}$ square faces in the d-hypercube.

Problem 5. Suppose there are L leaves and $n-L$ interior nodes, with n nodes total. Want to show $L > \frac{n}{2}$.

The handshake lemma says that

$$2 \cdot |E| = \sum_{v \in V} \deg(v) = \sum_{\substack{\text{leaves} \\ v}} 1 + \sum_{\substack{\text{non} \\ \text{leaves } v}} \deg(v)$$

$$= L \cdot 1 + \sum_{\substack{\text{non} \\ \text{leaves} \\ v}} \deg(v).$$

What is $|E|$? For a tree with n vertices, $|E| = n-1$.

(Proof by induction: base case: tree w/ 1 vertex & 0 edges. Then, for tree T w/ n vertices ~~edges~~, $T - v$ has $n-2$ edges by IH, so T has one more edge: $n-1$ edges).

So, $2(n-1) = L + \sum_{\substack{\text{non} \\ \text{leaves } v}} \deg(v)$. Since all non leaves have degree ≥ 3, this means

$$2(n-1) \geq L + 3(n-L) \Rightarrow 3n - 2L \leq 2n - 2$$

$\Rightarrow \quad n \leq 2L-2$

$\Rightarrow \frac{n+2}{2} \leq L$

$\Rightarrow L \geq \frac{n}{2}+1 > \frac{n}{2}$, as desired. $\square$.