HW 5 solutions.

Problem 1.

Sequential process:

1. Choose rank for 3 of a kind (13 ways)
2. choose suits for 3 of a kind ($\binom{4}{3} = 4$ ways)
3. Choose rank for pair (12 ways)
4. Choose suits for pair ($\binom{4}{2} = 6$ ways)

So, there are $13 \cdot 4 \cdot 12 \cdot 6$ full houses.

Problem 2.

Sequential process:

1. Choose 2 elements of $[5]$ $\binom{5}{2}$ ways
2. For each $x \in$ domain, choose 1 of the 2 elements to be $f(x)$ 2^5 ways

This count includes functions that map all x to only one value, which don't count — need 2 elements in the image. There are 2 such constant functions for each choice in step 1.

$\rightarrow$ $2^5 - 2$ ways to do step 2.

So, there are $\binom{5}{2} \cdot (2^5 - 2)$ functions $f: [5] \to [5]$ with $|f[[5]]| = 2$.

Problem 3.

3 cases:

(A) 2 TAs get apples, 2 TAs get oranges.

1. Choose which TAs get apples $\binom{4}{2}$ ways
2. Choose how many apples each gets
   - 20 stars, 1 bar, each gets at least 1 → $\binom{20-2+1}{1} = \binom{19}{1}$ ways
3. Choose how many oranges the other 2 TAs get
   - 10 stars, 1 bar, each gets at least 1 → $\binom{8+1}{1} = 9$ ways.

(B) 3 TAs get apples

1. Choose which TAs get apples $\binom{4}{3} = 4$ ways
2. Choose how many each gets
   $\binom{20-3+2}{2} = \binom{19}{2}$ ways
3. Other TA gets all 10 oranges 1 way

(C) 3 TAs get oranges.

1. Choose which get oranges $\binom{4}{3} = 4$ ways
2. Choose how many each gets
   $\binom{10-3+2}{2} = \binom{9}{2}$ ways
3. Other TA gets all 20 apples 1 way.

By addition principle, there are $\binom{4}{2} \cdot 19 \cdot 9 + \binom{4}{3} \cdot \binom{19}{2} + \binom{4}{3} \cdot \binom{9}{2}$ ways.

Problem 4. We construct a bijection from tidy to neat houses via $f(abcd) = ab(9-c)(9-d)$.

totality: $9-x$ is defined for any $x \in [9] \cup \{0\}$ ✓

existence: If $abcd$ is tidy, then $a+b = c+d$.

Want to show $a+b+9-c+9-d \stackrel{?}{=} 18$.

$$= a+b+18-c-d = c+d+18-c-d = 18.$$

So $ab(9-c)(9-d)$ is neat. ✓

uniqueness: $9-x$ is uniquely defined.

f is its own 2-sided inverse:

$$f(f(abcd)) = f(ab(9-c)(9-d))$$

$$= ab(9-(9-c))(9-(9-d))$$

$$= abcd$$

So f is a bijection.

Thus #neat houses = # tidy houses.

Problem 5

(a) Any district must have the k leftmost counties in the top row and $8-k$ leftmost counties in the bottom row, for some $k \in \{0,1,2,3,4,5,6,7,8\}$.

e.g. $k=0$:



$k=1$:



$k=2$:



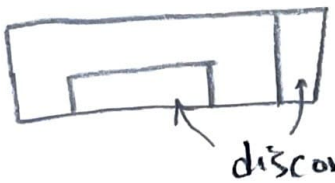
$k=7$:



$k=8$:



Any row where any non-leftmost counties are in the same district are not contiguous:



Note $k=0$ and $k=8$ give the same result. There are **8 ways**.

(b) 2-step process:

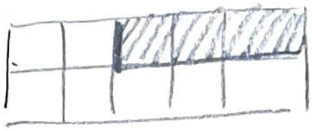
1. Choose sizes of the 2 districts. By stars & bars, where each must get at least 1 county, there are $\binom{2n-2+1}{1} = \binom{2n-1}{1} = 2n-1$ ways. Let the sizes be a and b.

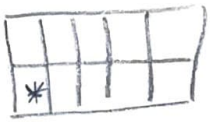
2. Choose a "start" point and let the a counties clockwise from the start point be the first district, and the other b be the 2nd district. $2n$ ways.

(For example: say the districts are size 3 and 7 ($n=5$), then if we choose this country as the start point:



then the districts will be



if we chose  then districts are )

This overcounts by a factor of 2: $a=3, b=7$,  gets the same districting as $a=7, b=3$, .

So, there are $\frac{(2n-1)\cdot(2n)}{2} = n(2n-1)$ ways.

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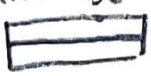
alternative answer: we do casework on the shape of the districts.

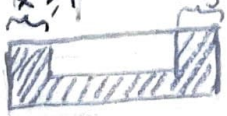
Case 1: , where top row has x, bottom row has y.

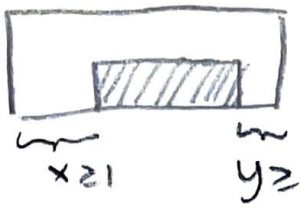
$n+1$ options for top, $n+1$ options for bottom

$0 \le x \le n$ $\quad$ $0 \le y \le n$.

$\Rightarrow (n+1)^2$ options. Both can't be $n+1$ or 0, so -2.

Also,  is counted twice, so -1.

Case 2: . Top is stars & bars where each section has at least one county. n stars, 2 bars $\rightarrow \binom{n-1}{2}$ ways.

Case 3: 

stars & bars again — $\binom{n-1}{2}$

Overall, this is $(n+1)^2 - 2 - 1 + \binom{n-1}{2} + \binom{n-1}{2}$

$= n^2 + 2n + 1 - 3 + \frac{(n-1)(n-2)}{2} + \frac{(n-1)(n-2)}{2}$

$= 2n^2 + 2n - 2 + -3n + 2 = 2n^2 - n$ ways.

(c). Let A_n be the # of ways to split Nsylvania into districts of size 2. $A_1 = 1$, $A_2 = 2 =$ .

For an arbitrary k,

k-sylvania can be tiled via either



or


2×1 $2\times(k-1)$. can be tiled A_{k-1} ways.

2 1×2's $2\times(k-2)$. Can be tiled A_{k-2} ways

So, $A_k = 1 \cdot A_{k-1} + 1 \cdot A_{k-2}$, by the addition principle.

So, A_k is just the k^{th} Fibbonacci #.