Problem Set 4 Solutions

Problem 1. $\{X : X \text{ is a student in COMS3203 in F26}\}$

Problem 2. If $|A| = n$, $|B| = m$, and $A \cap B = \emptyset$, then $|A \cup B| = m+n$.

Proof. Since $|A| = n$, there is a bijection $f: A \to [n]$.

Since $|B| = m$, there is a bijection $g: B \to [m]$.

Then, define a new bijection $h: A \cup B \to [m+n]$ via

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ n + g(x) & \text{if } x \in B. \end{cases}$$

h is well-defined.

- Totality: any $x \in A \cup B$ is in A or B; f is defined on $x \in A$; g is defined on x in B.
- Existence: $f(x) \in [n]$ and $[n] \subseteq [n+m]$
  $1 \leq g(x) \leq m$ so $n+g(x) \leq m+n$
  so $n+g(x) \in [n+m]$.
- Uniqueness: $A \cap B = \emptyset$, so no x is in both cases.

h is a bijection:

Surjectivity: Let $x \in [n+m]$. If $x > n$, then $g^{-1}(x-n)$ maps to x. If $x \leq n$, then $f^{-1}(x)$ maps to x.

~~h is a~~ injectivity: Suppose $h(x) = h(y)$.

If $h(x) > n$, then $n + g(x) = n + g(y)$ (since $f(x) \leq n$.)

$\Rightarrow g(x) = g(y)$

$\Rightarrow x = y$ since g is bijective.

if $h(x) \leq n$, then $f(x) = f(y)$ (since $n + g(x) \geq n+1 > n$).

$\Rightarrow x = y$ since f is a bijection.

Thus h is injective & surjective, so h is a bijection.

Since there is a bijection from $A \cup B$ to $[m+n]$,

$|A \cup B| = m+n$ by definition. $\square$.

Problem 3.

(a) $f: \mathbb{E} \to \mathbb{N}$ via $f(x) = x/2$.

(b) $g: \mathbb{E} \to \mathbb{Z}$ via $g = f \circ b$

(or, $g(n) = \begin{cases} n/4 & \text{if } n \text{ is a multiple of } 4 \\ -(n/2 - 1)/2 & \text{if } n \text{ is not a mult of } 4. \end{cases}$)

(c) $g^{-1}: \mathbb{Z} \to \mathbb{E}$ via $g^{-1}(x) = \begin{cases} 4x & \text{if } x > 0 \\ 2(-2x+1) & \text{if } x \leq 0. \end{cases}$

Let $x \in \mathbb{Z}$. $g(g^{-1}(x)) = ?$

If $x > 0$, $\hookrightarrow = g(4x) = \frac{4x}{4}$ since $4x$ is a mult of 4

$= x$. ✓

If $x \leq 0$, $g(g^{-1}(x)) = g(2(-2x+1)) = g(-4x+2)$.

$-4x+2$ is not a mult. of 4, so this is

$= -\left(\frac{-4x+2}{2} - 1\right)/2 = x$.

Let $n \in \mathbb{E}$. $g^{-1}(g(n)) = ?$

If $4 \mid n$, $g^{-1}(g(n)) = g^{-1}(n/4) = 4(n/4) = n$ since $n/4 > 0$.

If $4 \nmid n$, $g^{-1}(g(n)) = g^{-1}\left(-\frac{1}{2}\left(\frac{n}{2} - 1\right)\right)$. $-\frac{1}{2}\left(\frac{n}{2} - 1\right) \leq 0$, so this is

$= 2\left(-2\left(-\frac{1}{2}\left(\frac{n}{2} - 1\right)\right) + 1\right)$

$= n$. So g^{-1} is a 2-sided inverse.