Problem Set 3 solutions.

Problem 1.

Proposition: For sets A, X, $X \setminus (X \setminus A) = X \cap A$.

Proof. Let A, X be sets. Need to show: $y \in X \setminus (X \setminus A) \Leftrightarrow y \in X \cap A$

Suppose $y \in X \setminus (X \setminus A)$.

$\Leftrightarrow y \in X \wedge y \notin X \setminus A$ by def'n of set minus

$\Leftrightarrow y \in X \wedge \neg(y \in X \setminus A)$

$\Leftrightarrow y \in X \wedge \neg(y \in X \wedge y \notin A)$ DeMorgan's law

$\Leftrightarrow y \in X \wedge (y \notin X \vee y \in A)$

$\Leftrightarrow (y \in X \wedge y \notin X) \vee (y \in X \wedge y \in A)$

$\Leftrightarrow F \vee (y \in X \wedge y \in A)$

$\Leftrightarrow y \in X \cap A$. $\square$.

Problem 2.

Proposition: if $f: A \to A$ is idempotent ($f(f(x)) = f(x)\ \forall x$) then it is either the identity function $f(x) = x\ \forall x$ or not surjective.

Proof. We prove that if f is idempotent and surjective then it is the identity. Let $f: A \to A$ be idemp. & surj.

Since f is surjective, $\forall y \in A, \exists x \in A$ s.t. $f(x) = y$.

Then, $y = f(x) \Rightarrow f(y) = f(f(x))$ (applying f to both sides)

$\Rightarrow f(y) = f(x)$ (f is idempotent)

$\Rightarrow f(y) = y$ (since $y = f(x)$).

Thus, f is the identity. $\square$.

Problem 3.

(a) f_1 is not total. For example, $f_1(0000)$ is not defined.

(b) f_2 is well-defined.

Totality: Each $b \in B^*$ has length that is well-defined: $b \in B^* \Rightarrow b \in B_i$ for some $i \in \mathbb{N}$, and i is its length.

Existence: Lengths are all in $\mathbb{N}$, the codomain.

Uniqueness: Each $b \in B^*$ has exactly one length.

(c) f_3 is not unique. For example, $f_3(6)$ could be 011 or 0110 or 01100000.

(d) f_4 is well-defined.

Proof. (by induction).

We will prove the statement that $\forall n \in \mathbb{N}$, $\forall b \in B_n$, f_4 satisfies totality, existence, and uniqueness for b.

Base case: $n=1$. $B_1 = \{1, 0\}$.

$b = 1 \Rightarrow f(b) = 1$

$b = 0 \Rightarrow f(b) = 0$.

Totality: both cases are specified.

Existence: $1 \in \mathbb{Z}$ and $0 \in \mathbb{Z}$

Uniqueness: each b gets exactly 1 value.

Induction step. Assume that for some $k \in \mathbb{N}$, $\forall b \in B_k$, f_4 satisfies totality/existence/uniqueness for b.

Let $b = b_0 b_1 \dots b_k \in B_{k+1} = \{0,1\}^{k+1}$.

Totality: b_k is either 0 or 1, and both cases are covered. Let $b' = b_0 b_1 \dots b_{k-1} \in B_k$.

By the IH, f_4 is total for b', so f_4 is total for b.

Existence: By the IH, $f_4(b_0 \dots b_{k-1}) \in \mathbb{Z}$.

Then $f_4(b_0 \dots b_k) =$ either $2 \cdot$ an integer $+1$
or an integer.

In both cases, the result is still an integer.

Uniqueness: By the IH, $f_4(b_0 \dots b_{k-1})$ returns a unique value. Since any b falls in exactly one case, $f_4(b)$ is also unique.

By induction, f_4 is total for all $b \in B_n$ for any n. □

Problem 4.

Idea: how can we cover an infinitely-long interval with only a short interval from 0 to 1? One idea is to use $f(x) = \frac{1}{x}$, since $\frac{1}{x}$ hits any large number N using $\frac{1}{N} \in (0,1)$.

To cover both the positive & negative reals, we can split $(0,1)$ in half, using $(0, \frac{1}{2})$ to cover negative numbers and $(\frac{1}{2}, 1)$ to cover positive ones. (We'll use $\frac{1}{2}$ to cover 0.)

Another issue is that $f(x) = \frac{1}{x}$ doesn't cover y between 0 and 1, when $x \in (0,1)$. We can fix this by shifting down: $f(x) = \frac{1}{x} - 1$.

Combining these ideas, one bijection that works is:

$f: (0,1) \to \mathbb{R}$ via

$$f(x) = \begin{cases} \frac{1}{2(x-\frac{1}{2})} - 1 & \text{if } x > \frac{1}{2} \\ \frac{1}{2(x-\frac{1}{2})} + 1 & \text{if } x < \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2}. \end{cases}$$

Proof (that f is well-defined):

Totality. The cases are exhaustive, if $x \in (0,1)$ then $x > \frac{1}{2}$ or $x < \frac{1}{2}$ or $x = \frac{1}{2}$.

$2(x - \frac{1}{2})$ is never 0 when $x \neq \frac{1}{2}$, so we will never divide by 0.

Existence. When $x \neq \frac{1}{2}$, $\frac{1}{2(x-\frac{1}{2})} + 1 \in \mathbb{R}$ and $\frac{1}{2(x-\frac{1}{2})} - 1 \in \mathbb{R}$.

Also, $0 \in \mathbb{R}$. Thus the output is always in the codomain.

Uniqueness. The cases are disjoint: x cannot be both $> \frac{1}{2}$ and $< \frac{1}{2}$, etc.

Proof (that f is a bijection).

f is injective: Suppose $f(x) = f(y)$. Need to show $x = y$.

Case 1: $x > \frac{1}{2}$, $y > \frac{1}{2}$. Then $f(x) = f(y) \Rightarrow \frac{1}{2(x-\frac{1}{2})} - 1 = \frac{1}{2(y-\frac{1}{2})} - 1$

$\Rightarrow \frac{1}{2(x-\frac{1}{2})} = \frac{1}{2(y-\frac{1}{2})} \Rightarrow \frac{1}{x-\frac{1}{2}} = \frac{1}{y-\frac{1}{2}} \Rightarrow x - \frac{1}{2} = y - \frac{1}{2} \Rightarrow x = y$. ✓

Case 2: $x < \frac{1}{2}$, $y < \frac{1}{2}$. Then $f(x) = f(y) \Rightarrow \frac{1}{2(x-\frac{1}{2})} + 1 = \frac{1}{2(y-\frac{1}{2})} + 1$

$\Rightarrow 2(x - \frac{1}{2}) = 2(y - \frac{1}{2}) \Rightarrow x = y$. ✓

Case 3: $x = \frac{1}{2}$. $y = \frac{1}{2}$. Then $x = y$. ✓

Case 4: $x > \frac{1}{2}$, $y < \frac{1}{2}$. We need to show this case is impossible.

$x > \frac{1}{2} \Rightarrow$ first case, so $f(x) = \frac{1}{2(x-\frac{1}{2})} - 1$. Since $x < 1$,

$2(x - \frac{1}{2}) = 2x - 1 > 2(\frac{1}{2}) - 1 = 0$ and $2x - 1 < 2(1) - 1 = 1$.

So $0 < 2x - 1 < 1$, so $\frac{1}{2x-1} > 1$, and $\frac{1}{2x-1} - 1 > 0$.

$0 < y < \frac{1}{2} \Rightarrow f(y) = \frac{1}{2y-1} + 1$. Then $-1 < 2y - 1 < 0$, so $\frac{1}{2y-1} < -1$

and $\frac{1}{2y-1} + 1 < 0$.

So $f(x) > 0$ and $f(y) < 0$ and $f(x) = f(y)$, a contradiction.

Thus it can't be the case that $x > \frac{1}{2}$ and $y < \frac{1}{2}$.

Case 5: $x < \frac{1}{2}, y > \frac{1}{2}$. Same as case 4: this case is impossible.

Case 6: $x = \frac{1}{2}$, $0 < y < \frac{1}{2}$.

Then $f(x) = 0$, $f(y) = \frac{1}{2y-1} + 1$.

$$f(x) = f(y) \Rightarrow 0 = \frac{1}{2y-1} + 1$$

$$\Rightarrow 2y - 1 = -1$$

$$\Rightarrow 2y = 0 \Rightarrow y = 0, \text{ which contradicts } y > 0. \lightning$$

Case 7: $x = \frac{1}{2}$, $\frac{1}{2} < y < 1$.

$$f(x) = f(y) \Rightarrow 0 = \frac{1}{2y-1} - 1$$

$$\Rightarrow 2y - 1 = 1$$

$$\Rightarrow y = 1, \text{ which contradicts } y < 1.$$

Case 8 and 9: $y = \frac{1}{2}$, $x > \frac{1}{2}$ or $x < \frac{1}{2}$. Same as cases 6 and 7.

Thus, cases 1-3 are the only possible cases, and result in $x = y$. Thus f is injective.

f is surjective: Fix $r \in \mathbb{R}$.

- If $r = 0$, then $f(\frac{1}{2}) = 0$.
- If $r > 0$, then let $x = \frac{1}{2}(\frac{1}{r+1} + 1)$. Note $x > \frac{1}{2}$, since $\frac{1}{r+1} > 0$, and $x < 1$ since $\frac{1}{r+1} < 1 \Rightarrow \frac{1}{r+1} + 1 < 2 \Rightarrow \frac{1}{2}(\frac{1}{r+1} + 1) < 1$.

Then $f(x) = \frac{1}{2(\frac{1}{2}(\frac{1}{r+1}+1))-1} - 1 = \frac{1}{\frac{1}{r+1}+1-1} - 1$

$= r+1-1 = r.$

- If $r<0$, then let $x = \frac{1}{2}(\frac{1}{r-1}+1)$. Since $r<0$, $\frac{1}{r-1}<0$

so $x < \frac{1}{2}$, and $x>0$ since $r-1<-1 \Rightarrow \frac{1}{r-1} > -1$.

So $f(x) = \frac{1}{2(\frac{1}{2}(\frac{1}{r-1}+1))-1} + 1 = \frac{1}{\frac{1}{r-1}+1-1} + 1 = r-1+1 = r.$

Thus in all cases, $\exists x \in (0,1)$ where $f(x) = r$, so f is surjective.

Thus f is a bijection. $\square$.

Problem 5.

(a) $\mathbb{N} \subseteq \mathbb{R}$ is infinite and lonely.

$[0,1]$ is not lonely, and $[0,1] \subsetneq \mathbb{R}$.

(b). Any finite subset of the reals is lonely.

Proof. Let S be a finite subset of the reals.

Let $\text{diffs} = \{|x-y| ; x \in S, y \in S, x \neq y\}$.

Since S is finite, diffs is also finite. Let $\varepsilon = \frac{1}{2} \cdot \min(\text{diffs})$.

Let x be any element of S.

AFSOC $\exists y \in \mathbb{R}$ with $y \neq x$ where $\neg(|x-y| < \varepsilon \Rightarrow y \notin S)$.

This is equivalent to $|x-y| < \varepsilon$ and $y \in S$ (by $p \Rightarrow q \equiv \neg p \vee q$ and De Morgan's law).

If $y \in S$ and $|x-y| < \varepsilon = \frac{1}{2} \cdot \min(\text{diffs})$,

that means $|x-y| < \frac{1}{2} \cdot \min(\{|a-b| : a, b \in S, a \neq b\})$

and $x \in S$ and $y \in S$.

this means $\forall a, b \in S$, $a \neq b$,

$$|x-y| < \frac{1}{2} \cdot |a-b|.$$

This is a contradiction: let $a = x$, $b = y$; this would mean $|x-y| < \frac{1}{2}|x-y|$. ↯

Thus, $\nexists y \in \mathbb{R}$ with $y \neq x$ and $\neg(|x-y| < \varepsilon \Rightarrow y \notin S)$, so S is lonely. □.

(c) Let $S = \mathbb{N}$, and

$$T = \left\{ n + \tfrac{1}{n} : n \in \mathbb{N} \right\}.$$

Proposition: $S \cup T$ is not lonely.

That is: $\forall \varepsilon > 0$, $\exists x \in S \cup T$, $y \in \mathbb{R}$ with $y \neq x$ where $|x - y| < \varepsilon$ and $y \in S \cup T$.

Proof. Fix any $\varepsilon > 0$. For any $r \in \mathbb{R}$, let $\lceil r \rceil$ be the smallest integer larger than r.

Let $m = \lceil 1/\varepsilon \rceil + 1$. Since $\varepsilon > 0$, $m \in \mathbb{N}$, so $m \in S \cup T$.

Let $n = m + \frac{1}{m}$. Since $m \in \mathbb{N}$, $n \in T$ so $n \in S \cup T$.

Note $n \neq m$ since $\frac{1}{m} \neq 0$.

Finally, $|m - n| = \frac{1}{m} = \frac{1}{\lceil 1/\varepsilon \rceil + 1} < \frac{1}{\lceil 1/\varepsilon \rceil}$ (decreasing the denominator increases the value)

$\leq \frac{1}{1/\varepsilon}$ (taking the ceiling increases the value or keeps it the same)

$= \varepsilon$ (since $\varepsilon \neq 0$).

Thus $|m - n| < \varepsilon$, and $S \cup T$ is not lonely. $\square$.

(alternatively: $\frac{1}{m} < \varepsilon \iff \frac{1}{\lceil 1/\varepsilon \rceil + 1} < \varepsilon \iff \lceil 1/\varepsilon \rceil + 1 > \frac{1}{\varepsilon}$

Which is true because both $\lceil \cdot \rceil$ and $+1$ are nondecreasing.)