Problem Set 2 Solutions.

Problem 1.

(a) For $a, b \in \mathbb{Z}$, if $a^3 + ab^2 + b^3 = 0$, then a, b both even.

Proof. We prove the contrapositive:

$\neg$(a, b both even) $\Rightarrow a^3 + ab^2 + b^3 \neq 0$.

$\neg$(a even and b even) is a odd or b odd.

Case 1: a odd, b even. Then $\exists k, \ell \in \mathbb{Z}$ such that $\begin{cases} a = 2k+1 \\ b = 2\ell \end{cases}$

so $a^3 + ab^2 + b^3 = (2k+1)^3 + (2k+1)(2\ell)^2 + (2\ell)^3$

$= 8k^3 + 12k^2 + 6k + 1 + 8k\ell^2 + 4\ell^2 + 8\ell^3$

$= 2(4k^3 + 6k^2 + 3k + 4k\ell^2 + 2\ell^2 + 4\ell^3) + 1$

which is odd. So $a^3 + ab^2 + b^3 \neq 0$ since 0 is even. ✓

Case 2: a even, b odd.

Then a^3 even (multiplying even x anything = even)

and ab^2 even (same reason).

and b^3 odd (multiplying odds → odd).

so $a^3 + ab^2 + b^3$ is even + even + odd is odd, so $\neq 0$. ✓

Case 3: both odd.

Since multiplying odd #s gives odd #s,

a^3, ab^2, and b^3 are all odd.

Then the sum is odd + odd + odd = odd, so also $\neq 0$. ✓

Thus in all cases, $a^3 + ab^2 + b^3 \neq 0$. □.

(b) There is no rational x such that $x^3 + x + 1 = 0$.

Proof. AFSOC that $\exists x \in \mathbb{Q}$ where $x^3 + x + 1 = 0$.

Let $x = \frac{m}{n}$ for $m, n \in \mathbb{Z}$, in reduced form. m and n are not both even, since we're in reduced form.

Then $\left(\frac{m}{n}\right)^3 + \frac{m}{n} + 1 = 0 \Rightarrow m^3 + mn^2 + n^3 = 0$ by multiplying both sides by n^3.

By part (a), m and n must be both even. This is a contradiction. ↯

Thus, x cannot be rational. □.

Problem 2. If $n \in \mathbb{N}$ is odd, then n^2-1 is divisible by 8.

(a) Proof. Let $n \in \mathbb{N}$ be odd. Then $\exists k \in \mathbb{Z} : n = 2k+1$.

So $n^2-1 = (2k+1)^2-1 = 4k^2+4k+1-1 = 4k(k+1)$.

Either k or $k+1$ is even: {if k even, ✓
if k odd, $k = 2L+1$ for $L \in \mathbb{Z} \Rightarrow k+1 = 2L+2 = 2(L+1)$ which is even

So $k(k+1)$ is even : $\exists m \in \mathbb{Z}$, $k(k+1) = 2m$

So $n^2-1 = 4k(k+1) = 4(2m) = 8m$, as needed. □.

(b). Proof. we proceed by induction.

Let $P(n)$ be the predicate "$8 \mid n^2-1$".

We want to show $P(n)$ is true for all odd n.

Base case : $n=1$. $n^2-1 = 0$, and $8 \mid 0$ (because $0 = 8 \cdot 0$.)

Inductive step: Fix n (odd), and assume $8 \mid n^2-1$. We want to show that $P(n+2)$ is true: $8 \mid (n+2)^2-1$.

We expand:

$$(n+2)^2-1 = n^2+4n+4-1$$

$$= n^2-1+4(n+1)$$

By the inductive hypothesis, $n^2-1 = 8k$ for some $k \in \mathbb{Z}$. So

$$(n+2)^2-1 = 8k + 4(n+1).$$

Note $n+1$ is even since n is odd, so $n+1 = 2\ell$ for some $\ell \in \mathbb{Z}$. So

$$(n+2)^2-1 = 8k+4(2\ell) = 8(k+\ell).$$

Thus $(n+2)^2-1$ is divisible by 8.

By the principle of mathematical induction, $P(n)$ is true for all odd n. □.

Note: an alternative way would be to prove that $8 \mid (2k-1)^2-1 \ \forall k \in \mathbb{N}$

Problem 3.

Theorem. I can get n lollipops for $n = 3, 5, 6$, and all $n \geq 8$.

Proof. $3 =$ 1 pack of 3

$5 =$ 1 pack of 5

$6 =$ 2 packs of 3.

We prove the rest by (strong) induction. Let

$P(n) :=$ "I can get n lollipops using packs of 3 and 5."

Want to show $P(n)$ true for all $n \geq 8$.

Base cases: $P(8)$: 1 pack of 5, 1 pack of 3

$P(9)$: 3 packs of 3

$P(10)$: 2 packs of 5.

Inductive step. Let $n \geq 11$. Assume that $P(m)$ is true for all $8 \leq m < n$.

To buy n lollipops, I'll first buy $n-3$ lollipops.

Since $n-3 \geq 8$ and $n-3 < n$, $P(n-3)$ is true.

So, there's a x, y where x packs of 3 and y packs of 5 is $n-3$ lollipops total.

Then I can get n lollipops using $x+1$ packs of 3 and y packs of 5.

Thus $P(n)$ is true.

By induction, the theorem is true. □.

Problem 4.

Every $n \in \mathbb{N}$ can be expressed as the sum of distinct powers of 2.

Proof. (by induction).

Base case: $n=1$. $1 = 2^0$.

$n=2$. $2 = 2^1$.

Induction step: Assume that $\forall k \in \mathbb{N}$ where $1 \leq k \leq n$, k can be expressed as distinct powers of 2. (induction hypothesis).

Consider $n+1$.

- If $n+1$ is a power of 2, then $n+1 = 2^m$ for some $m \in \mathbb{N}$.

- If not, let 2^ℓ be the largest power of 2 less than $n+1$, where $\ell \geq 1$ since $n+1 \geq 2$. (because $n \geq 1$.)

by the I.H., $n+1-2^\ell < n$ can be expressed as distinct powers of 2: $n+1-2^\ell = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_m}$ for $a_1 \neq a_2 \neq \ldots \neq a_m$.

All of the a_i's must be $\neq \ell$, since 2^ℓ is the largest power of 2 less than $n+1$. (if one of the $a_i = \ell$, then $n+1-2^\ell \geq 2^\ell$, so $2^{\ell+1}$ is $< n+1$, and $2^{\ell+1}$ would be the largest power of 2 less than n.)

So $n+1 = 2^{a_1} + \ldots + 2^{a_m} + 2^\ell$ is $n+1$ expressed as distinct powers of 2.

By the principle of mathematical induction, all natural #s can be expressed as distinct powers of 2. ∎.

Problem 5.

Claim. If n is a multiple of 5, the player who goes next loses.

Proof.

Base case: $n=0$. The player who just went already won, so the next player has lost.

(Note: if you're not satisfied by this base case, you could also do $n=5$ as an additional base case, but the argument for $n=5$ is more or less the same as the inductive step.)

Induction step. Assume that, if there are $n=5k$ marbles left, the next player loses. $(k \in \mathbb{Z})$

Suppose there are $5k+5$ marbles.

- If Brandon takes 1 marble, Khine can take 4 marbles.
- " 2 " " 3 "
- " 3 " " 2 "
- " 4 " " 1 "

In all cases, Khine can make it such that there are $5k$ marbles when it is Brandon's turn again.

By the induction hypothesis, this is a losing situation for Brandon, and thus $5k+5$ marbles is as well.

The claim implies that when n is NOT a multiple of 5, the next player can win, by taking a number of marbles that makes the remainder a multiple of 5. Thus, the next player loses if and only if n is a multiple of 5.

QED.