HW1 Solutions.

Problem 1.

$\forall x \in \mathbb{Z}$, x is even $\Leftrightarrow$ $5x+3$ is odd.

<u>Proof</u>. First, we prove x even $\Rightarrow 5x+3$ odd.

Let $x \in \mathbb{Z}$ be even. Then, by def'n of even,

$\exists\, k \in \mathbb{Z}$ such that $x = 2k$.

Then, $5x+3 = 5(2k)+3 = 10k+3 = 2(5k+1)+1$.

Thus, $\exists\, L \in \mathbb{Z}$ where $5x+3 = 2L+1$, so $5x+3$ is odd. ✓

Now, we prove $5x+3$ odd $\Rightarrow x$ even.

By def'n of odd, $5x+3 = 2k+1$ for some $k \in \mathbb{Z}$.

$$\Rightarrow 5x = 2k-2$$

Assume for the sake of contradiction that x ~~ea~~ is odd.

Then $\exists\, m \in \mathbb{Z} : x = 2m+1$

$$\Rightarrow 5(2m+1) = 2k-2$$

$$\Rightarrow 10m+4+1 = 2k-2$$

$$\Rightarrow 2(5m+2)+1 = 2(k-1).$$

then this number is both even and odd, a contradiction.

$\lightning$.

Thus x must be even. $\square$.

Problem 2.

(a). $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x+y<0$.

Proof. Let x be any integer. We show that there is an integer y such that $x+y<0$.

If $x<0$, consider $y=0$. Then $x+y=x+0=x<0$ as needed.

If $x \geq 0$, consider $y=-x-1$. Then,

$$x+y = x-x-1 = -1 < 0, \text{ as needed.}$$

Thus in both cases there exists a y such that $x+y<0$. $\square$.

(Note: the first case actually isn't needed here. Why not?)

(b). We prove that the negation of the statement is true:

$\neg(\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ x+y<0)$

$\Leftrightarrow \forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ x+y \geq 0$.

Proof. Let x be any integer.

Consider $y=-x$. Then $x+y=0 \geq 0$, as needed. $\square$.

Problem 3. Let a, b be irrational and r be rational.

(a). $a+r$ must be irrational.

Proof.

AFSOC $a+r$ is rational.

Then $a+r = \frac{m}{n}$ for $m, n \in \mathbb{Z}$ and $r = \frac{p}{q}$ for $p, q \in \mathbb{Z}$.

Then $a = \frac{m}{n} - \frac{p}{q} = \frac{mq - np}{nq}$. But $mq - np \in \mathbb{Z}$ and $nq \in \mathbb{Z}$,

so a is rational. $\lightning$

Thus $a+r$ must be irrational. $\square$.

(b) $a+b$ can be rational.

Example: $a = \sqrt{2}$, $b = -\sqrt{2}$. $a+b = 0$, which is rational.

Proof that $-\sqrt{2}$ is irrational:

AFSOC $-\sqrt{2} = \frac{m}{n}$ for $m, n \in \mathbb{Z}$. Then $\sqrt{2} = \frac{-m}{n}$ is rational. $\lightning$.

Thus $-\sqrt{2}$ is irrational. $\square$.

(c) ar can be rational: consider r=0. Then ar=0, which is rational.

(d) ab can be rational.

Proof. Consider $a = \sqrt{2}$, $b = \sqrt{2}$. Then $ab = 2$, which is $\in \mathbb{Q}$.

$\square$.

(e) r^a can be rational.

Proof. Consider $r = 0$, $a = \sqrt{2}$.

Then $r^a = 0^{\sqrt{2}} = 0 \in \mathbb{Q}$. $\square$.

(f) (bonus). a^b can be rational.

Prop. $\exists$ a, b irrational where a^b is rational.

Proof. We do casework on whether $\sqrt{2}^{\sqrt{2}}$ is rational.

Case 1. If $\sqrt{2}^{\sqrt{2}}$ is rational, then let $a = b = \sqrt{2}$, and we have an example.

Case 2. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

Then $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational.

In both cases, there is an example of irrational a, b with $a^b \in \mathbb{Q}$, as desired. $\square$.

Note: this proof is what we call "non-constructive" – it doesn't actually give a concrete instance of a, b, it just shows that such an instance must exist.

Problem 4.

If $n \in \mathbb{N}$, then $\exists\ x \in \mathbb{Z}$ such that $n \leq x^2 \leq 2n$.

Proof. Let n be any natural number.

Let k be the smallest natural number where $k^2 \geq n$.

(Note that this k must exist, since there are many possible $\ell \in \mathbb{N}$ with $\ell^2 \geq n$, including at least $\ell = n$, $\ell = n+1$, $\ell = n+2, \ldots$ Let k be the smallest such ℓ.)

We need to show that $k^2 \leq 2n$.

- Since k is the smallest natural # with $k^2 \geq n$, we know that $(k-1)^2 < n$, so $(k-1)^2 + 1 \leq n$. $(*)$

Then,
$$
\begin{aligned}
k^2 &\leq k^2 + (k-2)^2 && \text{since } (k-2)^2 \geq 0 \text{ for any } k \\
&= 2((k-1)^2 + 1) && \text{rearranging} \\
&\leq 2n && \text{by } (*).
\end{aligned}
$$

Thus $k^2 \leq 2n$ as needed. $\square$.

Note: The middle bit of this proof is a bit opaque, isn't it? In a sense, it's because our scratch work was done "off screen".

A different but equivalent way would be to say, from $(*)$, that $2((k+1)^2+1) \leq 2n$. Now we want $k^2 \leq 2((k+1)^2+1)$

$\Leftrightarrow\ k^2 \leq 2k^2 - 4k + 4$

$\Leftrightarrow\ 0 \leq (m-2)^2$.

Thus

$k^2 \leq 2((k+1)^2+1) \leq 2n$.

Alternative proofs for #4.

Prop. $\forall n \in \mathbb{N}, \exists x \in \mathbb{N}$ where $n \le x^2 \le 2n$.

Proof #1. (by induction)

Base cases: $n=1$. Then $x=1$ works: $1 \le 1^2 \le 2$. ✓

$n=2$. Then $x=2$ works: $2 \le 2^2 \le 4$. ✓

Induction step. Let $n \ge 2$. Assume $\exists x \in \mathbb{N}$ such that $n \le x^2 \le 2n$. Since $n \ge 2$, $x \ge 2$.

Want to show: $\exists y \in \mathbb{N}$ such that $1+n \le y^2 \le 2(n+1)$.

(1) If $x^2 \ge n+1$, then $y=x$ works:

$n+1 \le x^2$, and $x^2 \le 2n$ (by IH)

$\le 2(n+1)$. ✓

(2) If $x^2 < n+1$, and $x^2 \ge n$ (by IH), then $x^2 = n$.

Then $y = x+1$ works:

$$n+1 \le y^2 \iff x^2+1 \le y^2 \quad \text{since } x^2=n$$
$$\iff x^2+1 \le (x+1)^2$$
$$\iff x^2+1 \le x^2+2x+1$$
$$\iff 0 \le 2x \quad \text{which is true whenever } x \ge 2.$$

And, $y^2 \le 2(n+1) \iff x^2+2x+1 \le 2x^2+2$

$$\iff 0 \le x^2-2x+1$$
$$\iff 0 \le (x-1)^2$$ which is also true whenever $x \ge 2$.

Thus by induction the claim is true for all $n \in \mathbb{N}$.

$\square$.

Proof. (#2 – an expression for x in terms of n).

Let x be the integer between $\sqrt{n}$ and $\sqrt{n}+1$. (that is, $\sqrt{n} \le x < \sqrt{n}+1$).

Need to show $n \le x^2 \le 2n$.

Since $\sqrt{n} \le x$, it is clear that $n \le x^2$.

To show $x^2 \le 2n$:

$$x^2 < (\sqrt{n}+1)^2 \quad \text{(by defn of } x\text{)}$$

$$= n + 2\sqrt{n} + 1.$$

When $n \geq 9$, $\sqrt{n} \le \frac{n}{3}$ and $1 \le \frac{n}{3}$, so this is

$$\le n + 2\left(\frac{n}{3}\right) + \frac{n}{3} = 2n, \text{ as needed.}$$

When $n < 9$, we manually show examples for x:

n	x	$2n$
1	1	2
2	2	4
3	2	6
4	2	8
5	3	10
6	3	12
7	3	14
8	3	16

$\square$.

Proof #3.

To show $\exists x \in \mathbb{N}$ where $n \le x^2 \le 2n$, we can equivalently show $\exists x \in \mathbb{N}$ where $\sqrt{n} \le x \le \sqrt{2n}$.

$$\iff \sqrt{n} \le x \le \sqrt{2}\sqrt{n}$$

$$\iff \sqrt{n} \le x \le \sqrt{n} + (\sqrt{2}-1)\sqrt{n}.$$

Notice that such an x is guaranteed to exist if $(\sqrt{2}-1)\sqrt{n} \ge 1$ (since there's always an integer between A and $A+1$ for any $A \in \mathbb{R}$).

$(\sqrt{2}-1)\sqrt{n} \ge 1$ whenever $n \ge \left(\frac{1}{\sqrt{2}-1}\right)^2 = 2.4142\ldots^2 = 5.82\ldots$

So we only need to manually check

~~$n=1 \Rightarrow x=1$ works. $1 \le 1^2 \le 2$~~

~~$n=2 \Rightarrow x=2$ works. $2 \le 2^2 \le 4$~~

for $n=1$ to 5.

n	x	$2n$
1	1	2
2	2	4
3	2	6
4	2	8
5	3	10

Thus it works for all n. $\square$.