Problem 1 (15 points)

Let p be the statement $\forall r \in \mathbb{R}_{\geq 0}, \exists z \in \mathbb{Z} : r \leq z \leq 2r$.

(a) Write the negation of p *without* using the word (or symbol) for "not."

$\exists r \in \mathbb{R}_{\geq 0} \quad \forall z \in \mathbb{Z}, \quad z < r \text{ or } z > 2r$

(b) Prove either p or its negation.

The negation is true.

Proof (by example):

Let $r = 1/4$. Then $2r = 1/2$.

Let z be any integer.

If $z \leq 0$, then $z < r$.

If $z \geq 1$, then $z > 2r$.

Thus $\forall z \in \mathbb{Z}, \quad z < r \text{ or } z > 2r$. □

Problem 2 (20 points)
Let $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Prove that f_{3n} is even for all $n \in \mathbb{N}$.

Proof. (by induction).

Base case: $n=1$. $f_3 = f_1 + f_2 = 1+1 = 2$ is even.

Induction step: Fix $n \in \mathbb{N}$. Assume that f_{3n} is even.

Then $f_{3(n+1)} = f_{3n+3}$

$= f_{3n+2} + f_{3n+1}$

$= (f_{3n+1} + f_{3n}) + (f_{3n+1}$ [illegible]

$= f_{3n} + 2 \cdot f_{3n+1}$.

By the IH, f_{3n} is even, and $2 \cdot f_{3n+1}$ is even, and even + even is even. So, $f_{3(n+1)}$ is even.

By induction, f_{3n} is even for all $n \in \mathbb{N}$. □.

Problem 3 (15 points)
Let A, B, C, D be sets.

(a) Prove that $(A \cap B) \setminus (C \cap D) \subseteq (A \setminus C) \cup (B \setminus D)$.

Suppose $x \in (A \cap B) \setminus (C \cap D)$.

Then $x \in A \wedge x \in B \wedge \neg(x \in C \wedge x \in D)$

$\Leftrightarrow x \in A \wedge x \in B \wedge (x \notin C \vee x \notin D)$

(by De Morgan's)

$\Leftrightarrow (x \in A \wedge x \in B \wedge x \notin C) \vee (x \in A \wedge x \in B \wedge x \notin D)$

(distributing $\wedge$ and $\vee$)

$\Rightarrow (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin D)$

$(P \wedge q \Rightarrow P)$

$\Leftrightarrow x \in (A \setminus C) \cup (B \setminus D)$. $\square$.

(b) Give an example to show that they are not necessarily equal.

$A = \{1, 2\}$ $\quad A \cap B = \{2\}$

$B = \{2, 3\}$ $\quad C \cap D = \emptyset$

$C = \{3\}$ $\quad \Rightarrow (A \cap B) \setminus (C \cap D) = \{2\}$

$D = \{4\}$

$A \setminus C = \{1, 2\}$

$B \setminus D = \{2, 3\}$

$(A \setminus C) \cup (B \setminus D) = \{1, 2, 3\}$.

Problem 4 (25 points)

Let $f : (0,\infty) \times (0,\infty) \to \mathbb{R} \times (0,\infty)$ via $f(x,y) = (x^2 - y^2, x + y)$.

(a) Prove that f is injective.

Proof. Suppose $f(x_1,y_1) = f(x_2,y_2)$.

$\Rightarrow (x_1^2-y_1^2, x_1+y_1) = (x_2^2-y_2^2, x_2+y_2)$

$\Rightarrow x_1^2-y_1^2 = x_2^2-y_2^2 \wedge x_1+y_1 = x_2+y_2$

$\Rightarrow (x_1-y_1)(x_1+y_1) = (x_2-y_2)(x_2+y_2) \wedge (x_1+y_1 = x_2+y_2)$

$\Rightarrow x_1-y_1 = x_2-y_2$ (divide both sides by $x_1+y_1 = x_2+y_2$)
$\wedge\, x_1+y_1 = x_2+y_2$ ($x_1,x_2,y_1,y_2 > 0$ so not div. by 0)

$\Rightarrow x_1-y_1+x_1+y_1 = x_2-y_2+x_2+y_2$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$.

Then, since $x_1+y_1 = x_2+y_2$
$\Rightarrow x_1+y_1 = x_1+y_2$
$\Rightarrow y_1 = y_2$.

Thus f is injective. □

(b) Show that f is not surjective.

There is no (x,y) where $f(x,y) = (100, 1)$.

AFSOC $x^2-y^2 = 100$, $x+y = 1$ for $x, y \in (0,\infty)$.

Since $y^2 \geq 0$ for any y,

$100 = x^2-y^2 \Rightarrow x^2 = 100+y^2 \Rightarrow x^2 \geq 100 \Rightarrow x \geq 10$ (since $x > 0$).

So $x+y \geq 10$. But $x+y = 1$, a contradiction. ↯

Thus, $\nexists (x,y) \in (0,\infty)\times(0,\infty)$ with $f(x,y) = (100,1)$. □.

Problem 5 (25 points)

For any given $n \in \mathbb{N}$, let S_n be the set of functions where n is a "fixed point," defined as

$$S_n = \{f : \mathbb{N} \to \mathbb{N} \text{ where } n = f(n)\}.$$

For example, $f(x) = x$, $g(x) = 4$, and $h(x) = (x-4)^2 + 4$ are all elements of S_4.

(a) Give an element of each of these sets:

$$a \in S_1, \qquad b \in \bigcap_{i=1}^{\infty} S_i, \qquad c \in F \setminus \left(\bigcup_{i=1}^{\infty} S_i\right)$$

where F is the set of all functions from $\mathbb{N} \to \mathbb{N}$.

$a: \mathbb{N} \to \mathbb{N}$ via $a(x) = x^2$ is in S_1.

$b: \mathbb{N} \to \mathbb{N}$ via $b(x) = x$ is in $\bigcap_{i=1}^{\infty} S_i$.

$c: \mathbb{N} \to \mathbb{N}$ via $c(x) = x+1$ is in $F \setminus \left(\bigcup_{i=1}^{\infty} S_i\right)$.

(b) Prove that for all $n \in \mathbb{N}$, if $f \in S_n$ and $g \in S_n$ then $f \circ g \in S_n$.

Proof. Let $n \in \mathbb{N}$. Then

$(f \circ g)(n) = f(g(n))$

$= f(n)$ since $g \in S_n$, so $g(n) = n$

$= n.$ since $f \in S_n$ so $f(n) = n$

Thus, $(f \circ g) \in S_n$.

(c) Prove that the converse is not true by showing that for any n, there are examples $f, g : \mathbb{N} \to \mathbb{N}$ where $f \circ g \in S_n$ but $f \notin S_n$ or $g \notin S_n$.

Let $n \in \mathbb{N}$, $g(x) = x+1$

$$f(x) = \begin{cases} x-1 & \text{if } x \geq 2 \\ 1 & \text{if } x = 1 \end{cases}.$$

For any n, $f(g(n)) = f(n+1) = \begin{cases} n+1-1 & \text{if } n+1 \geq 2 \\ 1 & \text{if } n+1 = 1 \end{cases}$

Since $n \in \mathbb{N}$, $n+1$ always ≥ 2, so this $= n+1-1 = n$.