Problem 1 (20 points)

For each of these questions, give an answer and a brief (1–2 sentence) explanation.

(a) Give a partition of the integers $\mathbb{Z}$ into 3 nonempty sets.

$\{0\}$, $\{x : x \in \mathbb{Z}, x > 0\}$, $\{x : x \in \mathbb{Z}, x < 0\}$.

The sets have no overlap. Every integer is either positive, negative, or 0.

(b) Is $[0, 1)$ finite, countably infinite, or uncountable?

It is uncountable - e.g. there's a injection from $\{0,1\}^{\mathbb{N}} \to [0,1)$ via $f(x_1x_2x_3\ldots) = 0.x_1x_2x_3\ldots$.

(c) How many ways can I distribute 30 oranges among 3 friends if each must get at least 5?

First distribute $3 \cdot 5 = 15$.

Remaining 15 + 2 bars $\to \binom{17}{2}$.

(d) Draw a 5-vertex graph with vertices of degree 2, 2, 3, 4, 4, or explain why it is not possible.

Not possible - handshake lemma says sum of degrees must be even but $2+2+3+4+4 = 15$ is odd.

Problem 2 (20 points)

Twelve (distinct) teddy bears line up to buy tickets to see *Paddington 3*. Andrew and Bartholomew got in a fight recently and refuse to be next to each other. How many ways can the bears arrange themselves in a line?

We count by subtracting out the orderings where A & B are next to each other.

There are 12! total orderings.

To choose orderings w/ A & B next to each other, :

1. Choose the places where A & B are (11 options)
2. Choose whether A or B is to the left (2 options)
3. Choose an order for the remaining 10 bears (10! options)

→ So, $11 \cdot 2 \cdot 10!$ options $(= 2 \cdot 11!$ options$)$.

So, orderings where A & B are not next to each other:

$12! - 2 \cdot 11!$ ways.

Problem 3 (20 points)
Using counting in two ways, prove that

$$\binom{m+n}{k} = \sum_{r=0}^{m} \binom{m}{r}\binom{n}{k-r}$$

Both sides count the # of ways to pick a team of k individuals from a population of m cats & n dogs.

LHS: Choose k from all $m+n$. $\binom{m+n}{k}$ ways.

RHS: partition by how many cats are on the team.
If there are r cats, then:

1. Choose r of the m cats $\binom{m}{r}$ ways
2. Choose $k-r$ of the n dogs $\binom{n}{k-r}$ ways.

So there are $\binom{m}{r}\binom{n}{k-r}$ ways to choose a team of k animals with r cats, by the mult. principle.

Any team has some # of cats between 0 and m, so this partitions the set. By addition principle,

$$\#\text{ ways} = \sum_{r=0}^{m} \binom{m}{r}\binom{n}{k-r}.$$

Thus, by counting in 2 ways, $\binom{m+n}{k} = \sum_{r=0}^{m} \binom{m}{r}\binom{n}{k-r}$.

□.

Problem 4 (20 points)

Let $n \geq 6$, and consider K_n, the complete graph with vertices $\{1, 2, \cdots n\}$.

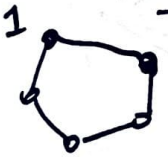
(a) How many length-5 *simple* paths are there in K_n?

You can go from any vertex to any other vertex, so we just have to pick 6 vertices with no repeats.

→ $n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4) \cdot (n-5)$

$= \dfrac{n!}{(n-6)!}$ (or, $\binom{n}{6} \cdot 6!$ also works)

(b) How many length-5 *simple cycles* are there in K_n that start (and end) at vertex 1?



↳ 5 vertices. Since it starts & ends at 1, we just need to choose the other 4.

→ $(n-1)(n-2)(n-3)(n-4)$ ways.

(or, $\binom{n-1}{4} \cdot 4!$.)

(c) How many length-5 paths (not necessarily simple) are there in K_n?

Repeats are allowed,

1. Choose 1st vertex (n ways)
2. Choose 2nd vertex (n-1 ways — can't be same as 1st)
3. 3rd " (n-1 ways)

⋮

6. Choose 6th vertex (n-1 ways)

So, $n \cdot (n-1)^5$ paths.

Problem 5 (20 points)

Let G be a graph with n vertices. Prove that if every vertex has degree $\geq \frac{n}{2}$, then G is connected.

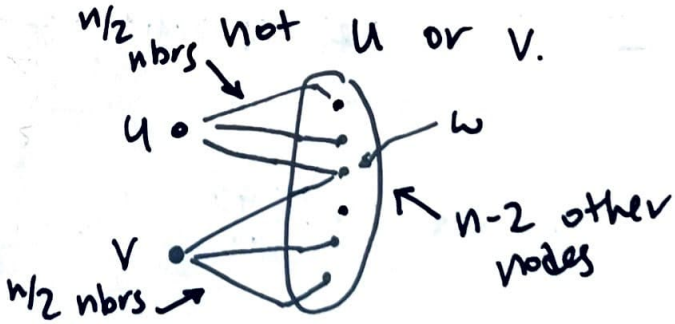
Hint: a graph is connected if every pair of vertices $u, v \in V$ have a path between them.

Proof. Let $u, v \in V$ be any pair of vertices.

Since all vertices have degree $\geq n/2$, u and v both have at least $\frac{n}{2}$ neighbors.

If u and v are neighbors, then there is a path between them: u, v.

If not, all $n/2$ neighbors are from somewhere else in the graph, i.e. the $n-2$ vertices that are



u and v, combined, have n edges (pigeons) going to $n-2$ other vertices (holes), so by PHP there must be a shared neighbor w. So, u and v are connected by a path, $u - w - v$.

Thus, G is connected. $\square$.