Discrete Mathematics

Midterm 1 Practice Exam

Name: _____

UNI: _____

Instructions.

- You have **75 minutes**.
- This exam is closed-book. No calculators or electronic devices.
- You may have up to one page of double-sided handwritten notes.
- There are 5 problems, worth 100 points in total.
- Write your solutions in the space provided. If you need more room, continue on the back of the page and say so clearly.

Question	Points	Score
1	15	
2	20	
3	15	
4	25	
5	25	
Total:	100	

Problem 1 (15 points)

Let p be the statement $\forall r \in \mathbb{R}_{\geq 0}, \exists z \in \mathbb{Z} : r \leq z \leq 2r$.

(a) Write the negation of p *without* using the word (or symbol) for "not."

(b) Prove either p or its negation.

Problem 2 (20 points)

Let $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Prove that f_{3n} is even for all $n \in \mathbb{N}$.

Problem 3 (15 points)

Let A, B, C, D be sets.

(a) Prove that $(A \cap B) \setminus (C \cap D) \subseteq (A \setminus C) \cup (B \setminus D)$.

(b) Give an example to show that they are not necessarily equal.

Problem 4 (25 points)

Let $f : (0, \infty) \times (0, \infty) \to \mathbb{R} \times (0, \infty)$ via $f(x, y) = (x^2 - y^2, x + y)$.

(a) Prove that f is injective.

(b) Show that f is not surjective.

Problem 5 (25 points)

For any given $n \in \mathbb{N}$, let S_n be the set of functions where n is a "fixed point," defined as

$$S_n = \{f : \mathbb{N} \to \mathbb{N} \text{ where } n = f(n)\}.$$

For example, $f(x) = x$, $g(x) = 4$, and $h(x) = (x-4)^2 + 4$ are all elements of S_4.

(a) Give an element of each of these sets:

$$a \in S_1, \qquad b \in \bigcap_{i=1}^{\infty} S_i, \qquad c \in F \setminus \left(\bigcup_{i=1}^{\infty} S_i \right)$$

where F is the set of all functions from $\mathbb{N} \to \mathbb{N}$.

(b) Prove that for all $n \in \mathbb{N}$, if $f \in S_n$ and $g \in S_n$ then $f \circ g \in S_n$.

(c) Prove that the converse is not true by showing that for any n, there are examples $f, g : \mathbb{N} \to \mathbb{N}$ where $f \circ g \in S_n$ but $f \notin S_n$ or $g \notin S_n$.