Website Supplement:
Genus Distributions of Graphs under Edge-Amalgamations

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Abstract

This document supplements a paper [PKG10a] submitted to *Ars Mathematicae Contemporanea*. It provides proof of the productions listed in Table 7.1 of that paper. It also provides a table of the partitioned genus distributions of the closed-end ladders L_0 through L_5.

1 Productions for Double-Edge-Rooted Graphs

We now derive the productions listed in Table 7.1 of [PKG10a], which are used for our application to closed-end ladders in §2. These productions are summarized in Table 1.1 for easy reference and for easy comparison to Table 7.1 of [PKG10a].

Theorem 1.1 *Let (G, e, d) and (H, g, f) be double-edge-rooted graphs, where all four roots have two 2-valent endpoints. Then the following productions apply when the fb-walks on both roots of the imbedding of G are distinct from each other and the imbedding of H is of type $\overrightarrow{dd''}$:*

$$dd_i^0(G) * \overrightarrow{dd''}_j(H) \longrightarrow 2dd_{i+j}^0(W) + 2ds_{i+j+1}^0(W) \tag{1.1}$$
$$ds_i^0(G) * \overrightarrow{dd''}_j(H) \longrightarrow 4dd_{i+j}^0(W) \tag{1.2}$$
$$sd_i^0(G) * \overrightarrow{dd''}_j(H) \longrightarrow 2sd_{i+j}^0(W) + 2ss_{i+j+1}^0(W) \tag{1.3}$$
$$ss_i^0(G) * \overrightarrow{dd''}_j(H) \longrightarrow 4sd_{i+j}^0(W) \tag{1.4}$$

Proof Productions (1.1) and (1.3) are both of form

$$xd_i^0(G,e,d) * \overrightarrow{dd''}_j(H,g,f) \;\longrightarrow\; 2xd_{i+j}^0(W,e,f) + 2xs_{i+j+1}^0(W,e,f)$$

where x is d in the former case and s in the latter case. Figure 1.1 shows how the fb-walks change in response to the breaking of fb-walks incident on the root-edges and recombining of the resulting strands. The first and last imbeddings show one less face as a result of amalgamation, while the middle two imbeddings show a decrease of three faces. The result follows from the Euler polyhedral equation.

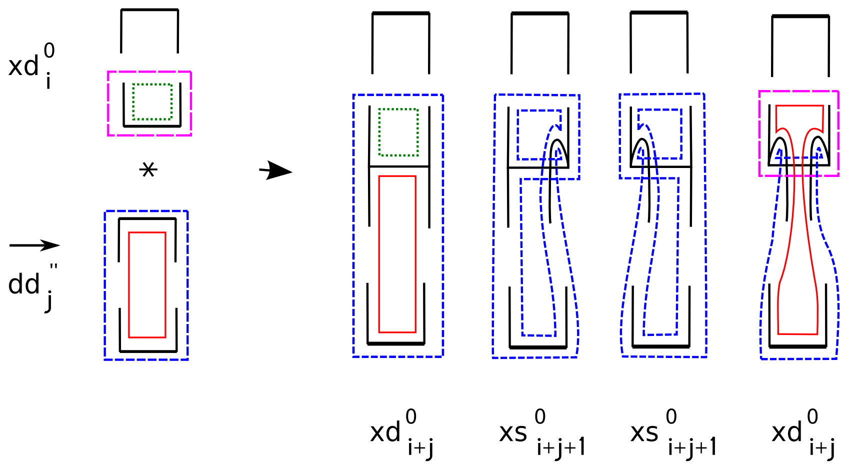

Figure 1.1: $xd_i^0(G) * dd''_j(H) \longrightarrow 2xd_{i+j}^0(W) + 2xs_{i+j+1}^0(W)$

In all cases, the fb-walks at edge e remain unaffected. Thus, the resulting imbedding for graph W has d or s for x, depending on whether there are two distinct fb-walks incident on edge e or only one in the graph G. The proofs of Productions (1.2) and (1.4) are very similar, and we leave these to the reader. $\diamondsuit$

Theorem 1.2 *Let (G,e,d) and (H,g,f) be double-edge-rooted graphs, where all four roots have two 2-valent endpoints. Then the following productions apply to the remaining cases where the fb-walks on each of the two roots of the imbedding of G are distinct and the imbedding of H is of type $\overrightarrow{dd''}$.*

$$\overline{dd'}_i(G) * \overrightarrow{dd''}_j(H) \;\longrightarrow\; dd_{i+j}^0(W) + \overline{dd'}_{i+j}(W) + 2\overrightarrow{ds'}_{i+j+1}(W) \qquad (1.5)$$

$$\widetilde{dd'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow dd^0_{i+j}(W) + \widetilde{dd'}_{i+j}(W) + 2\overleftarrow{ds'}_{i+j+1}(W) \tag{1.6}$$
$$\overrightarrow{dd'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow dd^0_{i+j}(W) + \overrightarrow{dd'}_{i+j}(W) + 2\overrightarrow{ds'}_{i+j+1}(W) \tag{1.7}$$
$$\overleftarrow{dd'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow dd^0_{i+j}(W) + \overleftarrow{dd'}_{i+j}(W) + 2\overleftarrow{ds'}_{i+j+1}(W) \tag{1.8}$$
$$\overrightarrow{dd''}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow \overline{dd'}_{i+j}(W) + \widetilde{dd'}_{i+j}(W) + 2ss^2_{i+j+1}(W) \tag{1.9}$$
$$\overleftarrow{dd''}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow \overrightarrow{dd'}_{i+j}(W) + \overleftarrow{dd'}_{i+j}(W) + 2ss^2_{i+j+1}(W) \tag{1.10}$$

Proof As before, we consider the way amalgamation on the root-edges in imbeddings of graphs G and H generates new fb-walks by recombining strands in the imbedding of the graph W. For the proof of Production (1.5), we look to Figure 1.2, which shows the new fb-walks of W as they arise from fb-walks in imbeddings of G and H.

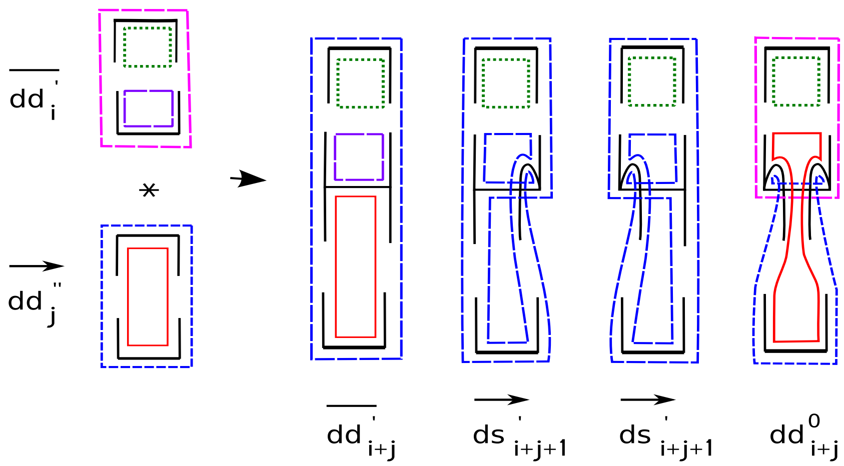

Figure 1.2: $\overline{dd'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow dd^0_{i+j}(W) + \overline{dd'}_{i+j}(W) + 2\overrightarrow{ds'}_{i+j+1}(W)$

Productions (1.6 - 1.8) also deal with amalgamation of a dd'-type imbedding of G with a $\overrightarrow{dd''}$-type imbedding of H. However, in each case the particular second-order partial of dd' causes different types of imbeddings to be generated. For example, Figure 1.3 highlights this contrast by providing the proof for Production (1.7).

Similarly, the picture proof of the Production (1.10) is given in Figure 1.4. The first and last imbedding of the graph W show one less face, while the second and the third imbedding of W show a decrease of 3 faces as all

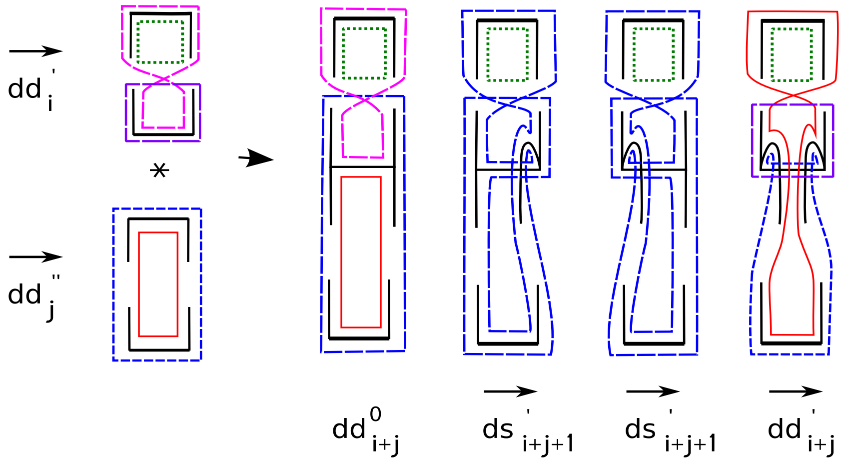

Figure 1.3: $\overrightarrow{dd'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow dd^0_{i+j}(W) + \overrightarrow{dd'}_{i+j}(W) + 2\overrightarrow{ds'}_{i+j+1}(W)$

the faces at root-edges d and f merge into a single face. The result follows. We leave the proofs of the remaining productions to the reader. $\diamondsuit$

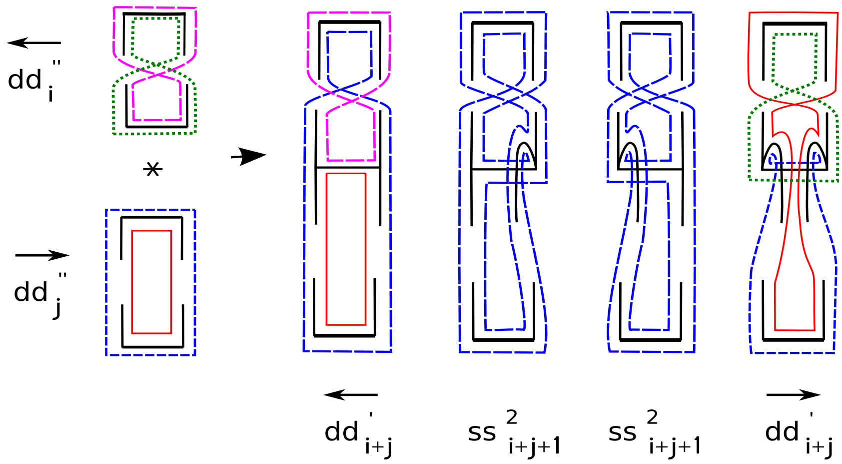

Figure 1.4: $\overleftarrow{dd''}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow \overrightarrow{dd'}_{i+j}(W) + \overleftarrow{dd'}_{i+j}(W) + 2ss^2_{i+j+1}(W)$

Theorem 1.3 *Let (G, e, d) and (H, g, f) be double-edge-rooted graphs, where all four roots have two 2-valent endpoints. Then the following productions apply when the imbedding of G is of type ds' or sd' and the imbedding of H is of type $\overrightarrow{dd''}$.*

$$\overrightarrow{ds'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow 2\overline{dd'}_{i+j}(W) + 2\overrightarrow{dd'}_{i+j}(W) \tag{1.11}$$

$$\overleftarrow{ds'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow 2\widetilde{dd'}_{i+j}(W) + 2\overleftarrow{dd'}_{i+j}(W) \tag{1.12}$$

$$\overrightarrow{sd'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow sd^0_{i+j}(W) + \overrightarrow{sd'}_{i+j}(W) + 2ss^1_{i+j+1}(W) \tag{1.13}$$

$$\overleftarrow{sd'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow sd^0_{i+j}(W) + \overleftarrow{sd'}_{i+j}(W) + 2ss^1_{i+j+1}(W) \tag{1.14}$$

Proof The proof for Production (1.11) follows from Figure 1.5. In all four cases that can arise as a consequence of amalgamation, the fb-walks incident at the root-edge d of graph G and the root-edge g of graph H break into strands that merge to yield one less face. Thus, the resulting genus of the imbedding of graph W is precisely the sum of the genera of imbeddings of G and H.

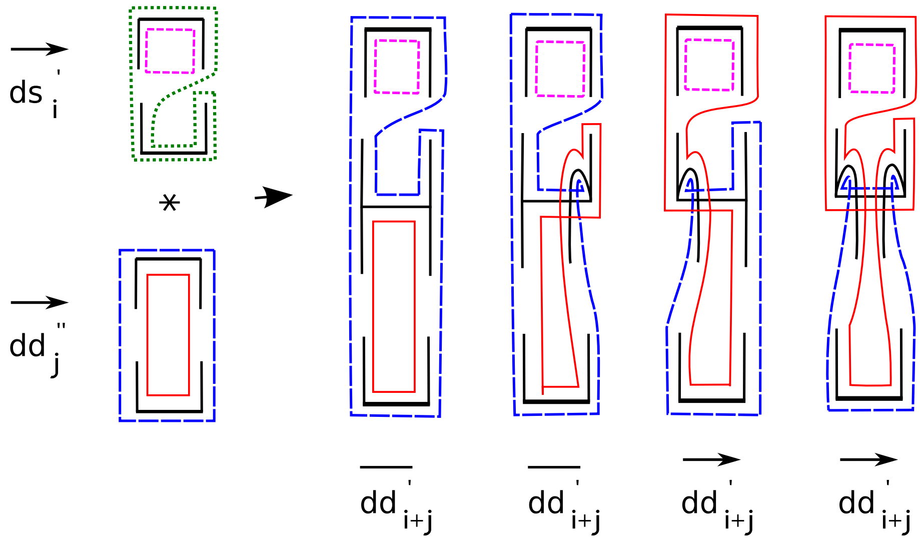

Figure 1.5: $\overrightarrow{ds'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow 2\overline{dd'}_{i+j}(W) + 2\overrightarrow{dd'}_{i+j}(W)$

The proof of Production (1.13) is similar. It follows by face-tracing, using as a model for $\overrightarrow{sd'}$ a 180° rotation of the model for $\overrightarrow{ds'}$ that we used in

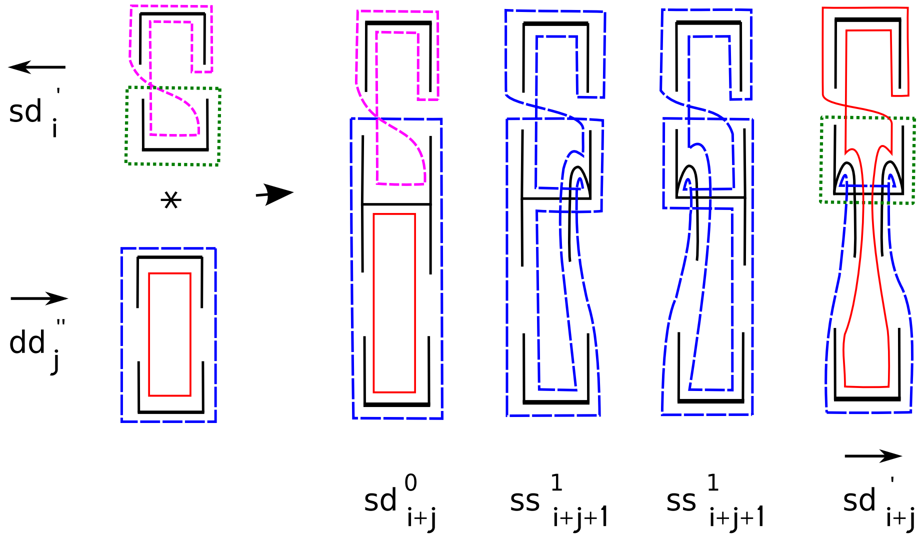

Figure 1.6: $\overleftarrow{sd'}_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow sd^0_{i+j}(W) + \overleftarrow{sd'}_{i+j}(W) + 2ss^1_{i+j+1}(W)$

Figure 1.5. We illustrate Production (1.14) by Figure 1.6. It is easy to use a 180° rotation of the model used for $\overleftarrow{sd'}$ and to use face-tracing to establish the proof of Production (1.12). ♢

Theorem 1.4 *Let (G, e, d) and (H, g, f) be double-edge-rooted graphs, where all four roots have two 2-valent endpoints. Then the following productions apply to all the remaining cases where the imbedding of G is of type ss and the imbedding of H is of type $\overrightarrow{dd''}$.*

$$ss^1_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow 2\overrightarrow{sd'}_{i+j}(W) + 2\overleftarrow{sd'}_{i+j}(W) \tag{1.15}$$

$$ss^2_i(G) * \overrightarrow{dd''}_j(H) \longrightarrow \overrightarrow{dd''}_{i+j}(W) + \overleftarrow{dd''}_{i+j}(W) + \overrightarrow{sd'}_{i+j}(W) + \overleftarrow{sd'}_{i+j}(W) \tag{1.16}$$

Proof The proofs of Productions (1.15) and (1.16) are clear from Figures 1.7 and 1.8, respectively.

For both productions, in all four cases, the genus of the induced imbedding surface of graph W is equal to the sum of the genera of the imbedding surfaces of the graphs G and H. However, the imbedding types of the graph W yielded by both productions are different. ♢

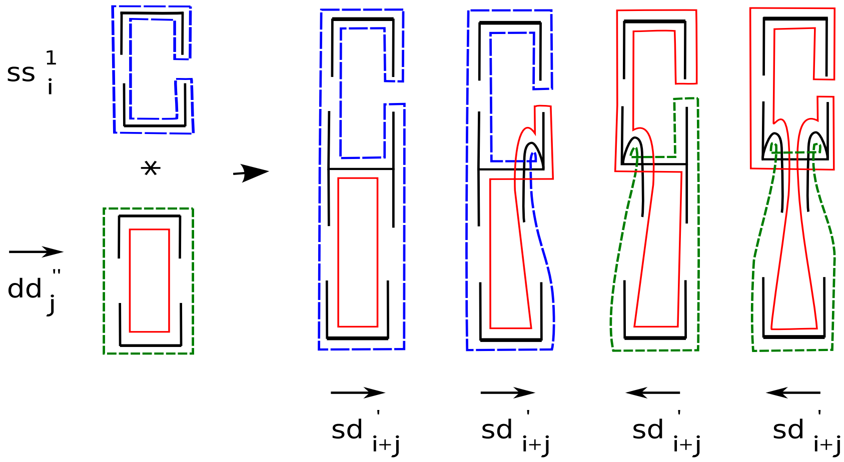

Figure 1.7: $ss_i^1(G) * \overrightarrow{dd''}_j(H) \longrightarrow 2\overrightarrow{sd'}_{i+j}(W) + 2\overleftarrow{sd'}_{i+j}(W)$

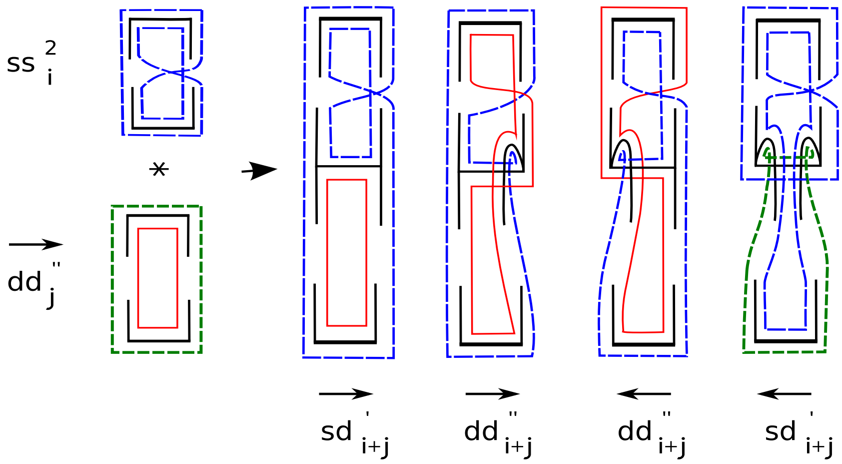

Figure 1.8: $ss_i^2(G) * \overrightarrow{dd''}_j(H) \longrightarrow \overrightarrow{dd''}_{i+j}(W) + \overleftarrow{dd''}_{i+j}(W)$
$+\overrightarrow{sd'}_{i+j}(W) + \overleftarrow{sd'}_{i+j}(W)$

We summarize the results of Theorems 1.1–1.4 in the following table, abbreviating the partials through omission of the graphs G, H and W.

Table 1.1: A subset of the productions for the edge-amalgamation $(G, e, d) * (H, g, f)$.

production	*reference*
$dd^0_i * \overrightarrow{dd''}_j \longrightarrow 2dd^0_{i+j} + 2ds^0_{i+j+1}$	(1.1)
$\overline{dd'}_i * \overrightarrow{dd''}_j \longrightarrow dd^0_{i+j} + \overline{dd'}_{i+j} + 2\overrightarrow{ds'}_{i+j+1}$	(1.5)
$\widetilde{dd'}_i * \overrightarrow{dd''}_j \longrightarrow dd^0_{i+j} + \widetilde{dd'}_{i+j} + 2\overleftarrow{ds'}_{i+j+1}$	(1.6)
$\overrightarrow{dd'}_i * \overrightarrow{dd''}_j \longrightarrow dd^0_{i+j} + \overrightarrow{dd'}_{i+j} + 2\overrightarrow{ds'}_{i+j+1}$	(1.7)
$\overleftarrow{dd'}_i * \overrightarrow{dd''}_j \longrightarrow dd^0_{i+j} + \overleftarrow{dd'}_{i+j} + 2\overleftarrow{ds'}_{i+j+1}$	(1.8)
$\overrightarrow{dd''}_i * \overrightarrow{dd''}_j \longrightarrow \overline{dd'}_{i+j} + \widetilde{dd'}_{i+j} + 2ss^2_{i+j+1}$	(1.9)
$\overleftarrow{dd''}_i * \overrightarrow{dd''}_j \longrightarrow \overrightarrow{dd'}_{i+j} + \overleftarrow{dd'}_{i+j} + 2ss^2_{i+j+1}$	(1.10)
$ds^0_i * \overrightarrow{dd''}_j \longrightarrow 4dd^0_{i+j}$	(1.2)
$\overrightarrow{ds'}_i * \overrightarrow{dd''}_j \longrightarrow 2\overline{dd'}_{i+j} + 2\overrightarrow{dd'}_{i+j}$	(1.11)
$\overleftarrow{ds'}_i * \overrightarrow{dd''}_j \longrightarrow 2\widetilde{dd'}_{i+j} + 2\overleftarrow{dd'}_{i+j}$	(1.12)
$sd^0_i * \overrightarrow{dd''}_j \longrightarrow 2sd^0_{i+j} + 2ss^0_{i+j+1}$	(1.3)
$\overrightarrow{sd'}_i * \overrightarrow{dd''}_j \longrightarrow sd^0_{i+j} + \overrightarrow{sd'}_{i+j} + 2ss^1_{i+j+1}$	(1.13)
$\overleftarrow{sd'}_i * \overrightarrow{dd''}_j \longrightarrow sd^0_{i+j} + \overleftarrow{sd'}_{i+j} + 2ss^1_{i+j+1}$	(1.14)
$ss^0_i * \overrightarrow{dd''}_j \longrightarrow 4sd^0_{i+j}$	(1.4)
$ss^1_i * \overrightarrow{dd''}_j \longrightarrow 2\overrightarrow{sd'}_{i+j} + 2\overleftarrow{sd'}_{i+j}$	(1.15)
$ss^2_i * \overrightarrow{dd''}_j \longrightarrow \overrightarrow{dd''}_{i+j} + \overleftarrow{dd''}_{i+j} + \overrightarrow{sd'}_{i+j} + \overleftarrow{sd'}_{i+j}$	(1.16)

The productions in Table 1.1 lead to the following theorem:

Theorem 1.5 *Let $(W, e, f) = (G, e, d) * (H, g, f)$, where each of the root-edges e, d, g, f has two 2-valent endpoints and the imbeddings of the graph H are of type $\overrightarrow{dd''}$. Then,*

$$dd^0_k(W) = \sum_{i=0}^{k} \left(2dd^0_i(G) + dd'_i(G) + 4ds^0_i(G)\right) \times \overrightarrow{dd''}_{k-i}(H) \quad (1.17)$$

$$\overline{dd'}_k(W) = \sum_{i=0}^{k} \left(\overline{dd'}_i(G) + \overrightarrow{dd''}_i(G) + 2\overrightarrow{ds'}_i(G)\right) \times \overrightarrow{dd''}_{k-i}(H) \quad (1.18)$$

$$\widetilde{dd'}_k(W) = \sum_{i=0}^{k} \left(\widetilde{dd'}_i(G) + \overrightarrow{dd''}_i(G) + 2\overleftarrow{ds'}_i(G)\right) \times \overrightarrow{dd''}_{k-i}(H) \tag{1.19}$$

$$\overrightarrow{dd'}_k(W) = \sum_{i=0}^{k} \left(\overrightarrow{dd'}_i(G) + \overleftarrow{dd''}_i(G) + 2\overrightarrow{ds'}_i(G)\right) \times \overrightarrow{dd''}_{k-i}(H) \tag{1.20}$$

$$\overleftarrow{dd'}_k(W) = \sum_{i=0}^{k} \left(\overleftarrow{dd'}_i(G) + \overleftarrow{dd''}_i(G) + 2\overleftarrow{ds'}_i(G)\right) \times \overrightarrow{dd''}_{k-i}(H) \tag{1.21}$$

$$\overrightarrow{dd''}_k(W) = \sum_{i=0}^{k} ss_i^2(G) \times \overrightarrow{dd''}_{k-i}(H) \tag{1.22}$$

$$\overleftarrow{dd''}_k(W) = \sum_{i=0}^{k} ss_i^2(G) \times \overrightarrow{dd''}_{k-i}(H) \tag{1.23}$$

$$ds_k^0(W) = \sum_{i=0}^{k-1} 2dd_i^0(G) \times \overrightarrow{dd''}_{k-1-i}(H) \tag{1.24}$$

$$\overrightarrow{ds'}_k(W) = \sum_{i=0}^{k-1} 2\left(\overline{dd'}_i(G) + \overrightarrow{dd'}_i(G)\right) \times \overrightarrow{dd''}_{k-1-i}(H) \tag{1.25}$$

$$\overleftarrow{ds'}_k(W) = \sum_{i=0}^{k-1} 2\left(\widetilde{dd'}_i(G) + \overleftarrow{dd'}_i(G)\right) \times \overrightarrow{dd''}_{k-1-i}(H) \tag{1.26}$$

$$sd_k^0(W) = \sum_{i=0}^{k} \left(2sd_i^0(G) + sd_i'(G) + 4ss_i^0(G)\right) \times \overrightarrow{dd''}_{k-i}(H) \tag{1.27}$$

$$\overrightarrow{sd'}_k(W) = \sum_{i=0}^{k} \left(\overrightarrow{sd'}_i(G) + 2ss_i^1(G) + ss_i^2(G)\right) \times \overrightarrow{dd''}_{k-i}(H) \tag{1.28}$$

$$\overleftarrow{sd'}_k(W) = \sum_{i=0}^{k} \left(\overleftarrow{sd'}_i(G) + 2ss_i^1(G) + ss_i^2(G)\right) \times \overrightarrow{dd''}_{k-i}(H) \tag{1.29}$$

$$ss_k^0(W) = \sum_{i=0}^{k-1} 2sd_i^0(G) \times \overrightarrow{dd''}_{k-1-i}(H) \tag{1.30}$$

$$ss_k^1(W) = \sum_{i=0}^{k-1} 2sd_i'(G) \times \overrightarrow{dd''}_{k-1-i}(H) \tag{1.31}$$

$$ss_k^2(W) \;=\; \sum_{i=0}^{k-1} 2dd_i''(G) \times \overrightarrow{dd''}_{k-1-i}(H) \tag{1.32}$$

Proof For instance, Production (1.1):

$$dd_i^0(G) * \overrightarrow{dd''}_j(H) \;\longrightarrow\; 2dd_{i+j}^0(W) + 2ds_{i+j+1}^0(W)$$

indicates that each dd^0-type imbedding of G on S_i when amalgamated with a $\overrightarrow{dd''}$-type imbedding of H on surface S_j, induces two imbeddings of W having type dd^0 on surface S_{i+j} and two of type ds^0 on surface S_{i+j+1}.

These contributions account for the term $\sum_{i=0}^{k} 2dd_i^0 \times dd''_{k-i}$ in Equation (1.17) and for the Equation (1.24). Taking into account all contributions made by the productions in Table 1.1, the result follows. $\diamondsuit$

2 Application: Closed-End Ladders

We illustrate some examples of closed-end-ladders in Figure 2.1.

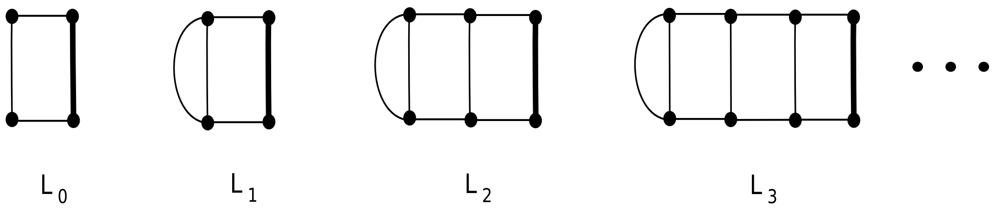

Figure 2.1: Closed-end ladders.

By face-tracing we know that all partials for L_0 are zero-valued except for $\overrightarrow{dd''}_0(L_0)$, whose value is 1. We now use the value of this partial and iteratively apply Theorem 1.5 to obtain Table 2.1. Observe that these values for $g_k(L_n)$ agree with the values first obtained by [FGS89].

Table 2.1: Double-root partials of L_n.

L_n	L_0	L_1		L_2		L_3			L_4			L_5			
k	0	0	1	0	1	0	1	2	0	1	2	0	1	2	3
dd_k^0	0	0	0	2	0	6	0	0	14	40	0	30	168	0	0
$\overline{dd'}_k$	0	1	0	1	0	1	6	0	1	10	0	1	14	56	0
$\widetilde{dd'}_k$	0	1	0	1	0	1	6	0	1	10	0	1	14	56	0
$\overrightarrow{dd'}_k$	0	0	0	0	0	0	6	0	0	10	0	0	14	56	0
$\overleftarrow{dd'}_k$	0	0	0	0	0	0	6	0	0	10	0	0	14	56	0
$\overrightarrow{dd''}_k$	1	0	0	0	2	0	0	0	0	0	8	0	0	0	0
$\overleftarrow{dd''}_k$	0	0	0	0	2	0	0	0	0	0	8	0	0	0	0
ds_k^0	0	0	0	0	0	0	4	0	0	12	0	0	28	80	0
$\overrightarrow{ds'}_k$	0	0	0	0	2	0	2	0	0	2	24	0	2	40	0
$\overleftarrow{ds'}_k$	0	0	0	0	2	0	2	0	0	2	24	0	2	40	0
sd_k^0	0	0	0	0	0	0	4	0	0	12	0	0	28	80	0
$\overrightarrow{sd'}_k$	0	0	0	0	2	0	2	0	0	2	24	0	2	40	0
$\overleftarrow{sd'}_k$	0	0	0	0	2	0	2	0	0	2	24	0	2	40	0
ss_k^0	0	0	0	0	0	0	0	0	0	0	8	0	0	24	0
ss_k^1	0	0	0	0	0	0	0	8	0	0	8	0	0	8	96
ss_k^2	0	0	2	0	0	0	0	8	0	0	0	0	0	0	32
g_k	1	2	2	4	12	8	40	16	16	112	128	32	288	576	128

References

[FGS89] M. L. Furst, J. L. Gross and R. Statman, Genus distribution for two classes of graphs, *J. Combin. Theory (B)* **46** (1989), 22–36.

[PKG10a] M. I. Poshni, I. F. Khan, and J. L. Gross, Genus distributions of graphs under edge-amalgamations, preprint, 2010.

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