ROOT GEOMETRY OF POLYNOMIAL SEQUENCES I: TYPE $(0,1)$

JONATHAN L. GROSS, TOUFIK MANSOUR, THOMAS W. TUCKER, AND DAVID G.L. WANG[†‡]

ABSTRACT. This paper is concerned with the distribution in the complex plane of the roots of a polynomial sequence $\{W_n(x)\}_{n\ge0}$ given by a recursion $W_n(x) = aW_{n-1}(x) + (bx + c)W_{n-2}(x)$, with $W_0(x) = 1$ and $W_1(x) = t(x - r)$, where $a > 0$, $b > 0$, and $c, t, r \in \mathbb{R}$. Our results include proof of the distinct-real-rootedness of every such polynomial $W_n(x)$, derivation of the best bound for the zero-set $\{x \mid W_n(x) = 0 \text{ for some } n \ge 1\}$, and determination of three precise limit points of this zero-set. Also, we give several applications from combinatorics and topological graph theory.

CONTENTS

Key words and phrases. genus distribution; real-rooted polynomial; recurrence; root geometry.
J.L. Gross is supported by Simons Foundation Grant #315001.
T.W. Tucker is supported by Simons Foundation Grant #317689.
D.G.L. Wang is supported by the Beijing Institute of Technology Research Fund Program for Young Scholars.

1. Introduction

Gian-Carlo Rota [26] has said of the ubiquity of roots of polynomials in combinatorics

"Disparate problems in combinatorics ... do have at least one common feature: their solution can be reduced to the problem of finding the roots of some polynomial or analytic function."

One such reduction is due to Newton's inequality, which implies that every real-rooted polynomial is log-concave. As observed by Brenti [1, 2], polynomials that arise from combinatorial problems often turn out to be real-rooted.

Given a sequence $\{W_n(x)\}_{n\ge0}$ of polynomials, we refer to the distribution of the set of zeros, taken over all n, as the *root geometry* of that sequence. General information for the root geometry of polynomials, especially the geometry of non-real roots, is given by Marden [19]; see also [21, 24].

This research arose during efforts by the present authors to affirm a quarter-century old conjecture (abbr. the LCGD conjecture) that the genus distribution (or genus polynomial, equivalently) of every graph is log-concave [8]. Although it was conjectured by Stahl [28] that genus polynomials are real-rooted, Chen and Liu [4] proved otherwise by the counterexample of iterated 4-wheels. Subsequently, various genus polynomials have been shown to have complex roots. Of course, this separates the problem of determining which graphs have real-rooted genus polynomials from trying to prove the LCGD conjecture. The log-concavity of genus distribution of the iterated 4-wheels is confirmed recently by the authors, see [13].

After unexpected success [12] in proving the real-rootedness of the genus polynomials of iterated claws, we attempted the real-rootedness of genus polynomials for iterated 3-wheels [23]. The *iterated 3-wheel* W_3^n is the graph obtained from the cartesian product $C_3\Box P_{n+1}$, where P_k is a path graph with k vertices, by contracting a 3-cycle C_3 at one end of the product to a single vertex. By a preprocess of normalization, we transformed the problem equivalently into the following conjecture.

Conjecture 1.1. Let $W_0(x) = 1/27$, $W_1(x) = 1 + 7x$, $W_2(x) = 1 + 139x + 1120x^2 + 468x^3$, and

$$W_n(x) \;=\; (1 + 144x)W_{n-1}(x) + 54x(2 - 29x + 306x^2)W_{n-2}(x) - 5832x^3(1 - 11x)W_{n-3}(x),$$

for $n \ge 3$. Then each of the polynomials $W_n(x)$ is real-rooted.

Real-rootedness of the genus polynomials of iterated 3-wheels W_3^n was confirmed by brute force computation for all $n \le 220$. The complications encountered led us to consider the more general problem for polynomial sequences defined by a general linear recurrence of degree 3, with polynomial coefficients. As one may imagine, the difficulty did not decrease. This led us to some recurrences of degree 2. In particular, let $W_n(x)$ be a sequence of polynomials satisfying the recursion

$$W_n(x) \;=\; A(x)W_{n-1}(x) + B(x)W_{n-2}(x) \tag{1.1}$$

for $n \ge 2$, where $A(x)$ and $B(x)$ are polynomials, $W_0(x)$ is a constant, and $W_1(x)$ is a linear polynomial. When the polynomials $A(x)$ and $B(x)$ have degrees k and ℓ, respectively, we call the sequence $\{W_n(x)\}$ defined by Rec. (1.1) *a recursive polynomial sequence of type* (k, ℓ).

Classical bounds on the roots of a polynomial are given in terms of its coefficients. Examples include the Fujiwara bound [7], the Cauchy bound [3], and the Hirst-Macey bound [16]. More bounds and also some background are given by Rahman and Schmeisser [25], where the reader may also find, for instance, Rouché's theorem, Landau's inequality, and the Laguerre-Samuelson inequality, subject to bounding the roots of a polynomial. Conversely, the real-rooted polynomials with all roots in a prescribed interval have been characterized in terms of positive semi-definiteness of related Hankel matrices; see Lasserre [18].

This paper is primarily concerned with the root geometry of a sequence of recursive polynomials of type $(0, 1)$.

2. Main Results and Examples

As a preliminary, we consider a recursive polynomial sequence of type $(0, 0)$, that is, one in which the polynomials $A(x)$ and $B(x)$ are constants, A and B. This serves as a bridge to considering a recursive sequence of polynomials of types in which $A(x)$ and $B(x)$ have other degree combinations.

Lemma 2.1. *Let $A, B \in \mathbb{R}$ with $A \neq 0$. Let $\{W_n\}_{n\geq 0}$ be a sequence of real numbers satisfying the initial condition $W_0 = 1$ and the recursion $W_n = AW_{n-1} + BW_{n-2}$. Writing*

$$\Delta = A^2 + 4B \qquad \text{and} \qquad g^{\pm} = \frac{2W_1 - A \pm \sqrt{\Delta}}{2},$$

we have

$$W_n = \begin{cases} \left(1 + \dfrac{n(2W_1 - A)}{A}\right)\left(\dfrac{A}{2}\right)^n, & \text{if } \Delta = 0; \\ \dfrac{g^+(A + \sqrt{\Delta}\,)^n - g^-(A - \sqrt{\Delta}\,)^n}{2^n\sqrt{\Delta}}, & \text{if } \Delta \neq 0. \end{cases} \tag{2.1}$$

In particular, if $Re^{i\theta}$ is the polar representation of $A + \sqrt{\Delta}$, then we have

$$W_n = \left(\frac{R}{2}\right)^n \left(\cos n\theta + \frac{\sin n\theta}{\sqrt{-\Delta}}\right), \qquad \text{if } \Delta < 0. \tag{2.2}$$

Proof. The solution (2.1) to Rec. (1.1) can be found in elementary textbooks; for more extensive discussion, see Kocic and Ladas [17]. Note that when $A + \sqrt{\Delta} = Re^{i\theta}$, we have $A - \sqrt{\Delta} = Re^{-i\theta}$, since $\sqrt{\Delta}$ is either purely real or purely imaginary. Then, Eq. (2.2) can be obtained from Eq. (2.1) directly. □

For instance, the Fibonacci sequence $\{f_n\}_{n\geq 0}$ is defined by the recursion $f_n = f_{n-1} + f_{n-2}$, with $f_0 = f_1 = 1$. With $A = B = W_1 = 1$ (hence, $\Delta = 5$ and $g^{\pm} = (1 \pm \sqrt{5})/2$), Lemma 2.1 gives Binet's formula, as expected:

$$W_n = \frac{(g^+)^{n+1} - (g^-)^{n+1}}{\sqrt{5}}.$$

Thus, we see how Lemma 2.1 creates conditions for recursive sequences of type $(0, 0)$, under which the root geometry problem becomes easy.

2.1. **Main result.** The aim of this paper is to describe the root geometry of all recursive polynomial sequences of type $(0,1)$. In order to formulate the main results of this paper, we use the following terminology.

Definition 2.2. The *zero-set* of a polynomial is defined to be the set of all its roots. It is said to be *distinct-real-rooted* if all its roots are distinct and real.

Definition 2.3. Let s be a positive integer, and let $t \in \{s-1,\, s\}$. Let $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$ be ordered sets of real numbers. We say that the set X *interlaces the set* Y *from both sides*, denoted $X \bowtie Y$, if $t = s-1$ and

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{s-1} < y_t < x_s. \tag{2.3}$$

Note that the bow-tie symbol $\bowtie$ consists of a "times" symbol $\times$ in the middle and a bar at each side. The left (resp., right) bar indicates that the smallest (resp., largest) number in Ineq. (2.3) is from the set X. We say that the set X *interlaces* Y *from the right*, denoted $X \rtimes Y$, if either $X \bowtie Y$, or

$$y_1 \;<\; x_1 \;<\; y_2 \;<\; x_2 \;<\; \cdots \;<\; x_{s-1} \;<\; y_t \;<\; x_s, \qquad \text{where } t = s. \tag{2.4}$$

Here the bar to the right of the "times" symbol $\times$ within the symbol $\rtimes$ means that the largest number in Ineq. (2.4) is from X. We observe that any set consisting of a single real number interlaces the empty set.

For any integers $m \le n$, we denote the set $\{m,\, m+1, \ldots,\, n\}$ by $[m, n]$. Moreover, when $m = 1$, we may denote the set $[1, n]$ by $[n]$. Lemma 2.4 presents some essential consequences of the interlacing property.

Lemma 2.4. *Let $f(x)$ and $g(x)$ be polynomials with zero-sets X and Y respectively. Let $\beta \in \mathbb{R}$, and let*

$$X' = X \cap (-\infty,\, \beta) = \{x_1,\, x_2,\, \ldots,\, x_p\} \qquad \text{and} \qquad Y' = Y \cap (-\infty,\, \beta) = \{y_1,\, y_2,\, \ldots,\, y_q\}$$

be two ordered sets such that $X' \rtimes Y'$. Let $x_0 = y_0 = -\infty$ and $y_{q+1} = \beta$.

- *If $f(\beta) \neq 0$, then we have*

$$f(y_j)f(\beta)(-1)^{q-j} < 0 \qquad \text{for all } j \in [q+1-p,\, q+1]; \tag{2.5}$$

- *If $g(\beta) \neq 0$, then we have*

$$g(x_i)g(\beta)(-1)^{p-i} > 0 \qquad \text{for all } i \in [p-q,\, p]. \tag{2.6}$$

Proof. See Appendix A. □

Notation 2.5. For any sequence $\{x_n\}$ of real numbers, we write $x_n \searrow x$ if x_n converges to the number x decreasingly, and we write $x_n \nearrow x$ if x_n converges to the number x increasingly.

Our main result, Theorem 2.6, concerns a polynomial sequence $W_n(x)$ of type $(0,1)$ in which $A(x) = a$ and $B(x) = bx + c$, with $ab \neq 0$ and $c \in \mathbb{R}$.

Theorem 2.6. *Let $\{W_n(x)\}_{n\ge 0}$ be the polynomial sequence defined by the recursion*

$$W_n(x) \;=\; aW_{n-1}(x) + (bx+c)W_{n-2}(x), \tag{2.7}$$

with $W_0(x) = 1$ and $W_1(x) = t(x-r)$, where $a, b, t > 0$, $c, r \in \mathbb{R}$, and $r \neq -c/b$. Then the polynomial $W_n(x)$ has degree $d_n = \lfloor (n+1)/2 \rfloor$ and is distinct-real-rooted. Moreover, let

$$R_n \;=\; \{\xi_{n,1},\, \xi_{n,2},\, \ldots,\, \xi_{n,d_n}\}$$

be the zero-set of $W_n(x)$*, where* $\xi_{n,1} < \xi_{n,2} < \cdots < \xi_{n,\,d_n}$*. Let* $R'_n = R_n \setminus \{\xi_{n,\,d_n}\}$*. Using the notations*

$$x^* = -\frac{4c+a^2}{4b}, \qquad r^* = x^* - \frac{a}{2t}, \qquad \text{and} \qquad y^* = r + \frac{(at+b) - \sqrt{(at+b)^2 + 4t^2(br+c)}}{2t^2},$$

we have the following conclusions:

(i) *If* $r \in (-\infty, r^*]$*, then* $R_{n+1} \rtimes R_n$ *and* $R_{n+2} \bowtie R_n$ *for* $n \ge 1$*;* $\xi_{n,\,d_n - i} \nearrow x^*$ *for any* $i \ge 0$*.*

(ii) *If* $r \in (r^*, -c/b)$ *then* $R_{n+1} \rtimes R_n$ *and* $R_{n+2} \bowtie R_n$ *for* $n \ge 1$*;* $\xi_{n,\,d_n - i} \nearrow x^*$ *for any* $i \ge 1$*; and* $\xi_{n,d_n} \nearrow y^*$ *with* $x^* < y^*$*.*

(iii) *If* $r \in (-c/b, +\infty)$ *then* $R'_{n+1} \rtimes R'_n$ *and* $R'_{n+2} \bowtie R'_n$ *for* $n \ge 3$*;* $\xi_{n,d_n - i} \nearrow x^*$ *for any* $i \ge 1$*;* $\xi_{2n,d_{2n}} \nearrow y^*$ *and* $\xi_{2n-1,d_{2n-1}} \searrow y^*$ *with* $x^* < -c/b < \xi_{2,d_2} < y^* < r$*.*

The best bounds for the set $\cup_{n\ge 1} R_n$ *are, in these three respective cases,* $(-\infty,\, x^*)$*,* $(-\infty,\, y^*)$ *and* $(-\infty,\, r)$*. Furthermore, the sequence* $\xi_{n,i}$ *converges to* $-\infty$ *for any fixed* $i \ge 1$*.*

We observe that in the statement of Theorem 2.6, the limit point x^* does not depend on the initial polynomial $W_1(x)$, as long as the polynomial $W_1(x)$ is linear, and furthermore, no root lies in the interval $(x^*,\, -c/b)$ for case (iii).

In fact, when $W_1(x) = t(x-r)$, we can always normalize the polynomials by the linear transformation

$$\overline{W}_n(x) \;=\; W_n(x/t + r),$$

whose root geometry differs from that of the sequence $W_n(x)$ only by magnification and translation. From Rec. (2.7), one may infer that

$$\overline{W}_n(x) \;=\; a\overline{W}_{n-1}(x) + (bx/t + br + c)\overline{W}_{n-2}(x),$$

with $\overline{W}_0(x) = 1$ and $\overline{W}_1(x) = x$. Therefore, we can suppose that $W_1(x) = x$ from the beginning.

Suppose $W_1(x) = x$. Now, if $c = 0$, then the number 0 is a root of every polynomial $W_n(x)$. In this circumstance, we have

$$W_2(x) = (a+b)x \qquad \text{and} \qquad W_3(x) = aW_2(x) + bx^2.$$

Consider the polynomials

$$\widetilde{W}_n(x) \;=\; \frac{W_{n+2}((a+b)(x-a)/b)}{W_2((a+b)(x-a)/b)}.$$

From Rec. (2.7), we infer that

$$\widetilde{W}_n(x) \;=\; a\widetilde{W}_{n-1}(x) + (a+b)(x-a)\widetilde{W}_{n-2}(x), \tag{2.8}$$

with $\widetilde{W}_0(x) = 1$ and

$$\widetilde{W}_1(x) \;=\; \frac{W_3((a+b)(x-a)/b)}{W_2((a+b)(x-a)/b)} \;=\; a + \frac{b}{a+b}\cdot\frac{(a+b)(x-a)}{b} \;=\; x.$$

Since the constant term of the coefficient polynomial $(a+b)(x-a)$ in Rec. (2.8) is $-a(a+b) \neq 0$, we can suppose that $c \neq 0$ from the beginning.

To give a proof of Theorem 2.6, we state its "normalized" version as Theorem 2.8, in which we restrict $W_1(x) = x$ and $c \neq 0$. As will be seen, Theorem 2.8 implies Theorem 2.6 conversely. The following notion of $(0,1)$-sequence of polynomials is the key object we will study; see Sections 3 to 5.

Definition 2.7. Let $\{W_n(x)\}_{n\ge 0}$ be the polynomial sequence defined recursively by

$$W_n(x) \;=\; aW_{n-1}(x) + (bx+c)W_{n-2}(x),$$

with $W_0(x)=1$ and $W_1(x)=x$, where $a,b>0$ and $c\ne 0$. In this context, we say $\{W_n(x)\}_{n\ge 0}$ is a $(0,1)$-*sequence of polynomials*. It is clear that $(0,1)$-sequence of polynomials is the particular case studied in Theorem 2.6 for $t=1$ and $r=0$.

Theorem 2.8. *Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials. Then the polynomial $W_n(x)$ (of degree $d_n=\lfloor (n+1)/2\rfloor$) is distinct-real-rooted. Let*

$$x^* = -\frac{4c+a^2}{4b}, \qquad r^* = x^* - \frac{a}{2}, \qquad \textit{and} \qquad y^* = \frac{a+b-\sqrt{(a+b)^2+4c}}{2}. \tag{2.9}$$

Let $R_n=\{\xi_{n,1},\,\xi_{n,2},\,\ldots,\,\xi_{n,d_n}\}$ be the ordered zero-set of $W_n(x)$.

- (i) *If $r^*\ge 0$, then $R_{n+1}\rtimes R_n$ and $R_{n+2}\bowtie R_n$ for $n\ge 1$; $\xi_{n,\,d_n-i}\nearrow x^*$ for any fixed $i\ge 0$.*
- (ii) *If $0\in(r^*,-c/b)$ then $R_{n+1}\rtimes R_n$ and $R_{n+2}\bowtie R_n$ for $n\ge 1$; $\xi_{n,\,d_n-i}\nearrow x^*$ for any fixed $i\ge 1$; and $\xi_{n,d_n}\nearrow y^*$ with $x^*<y^*$.*
- (iii) *If $c>0$ then $R'_{n+1}\rtimes R'_n$ and $R'_{n+2}\bowtie R'_n$ for $n\ge 3$; $\xi_{n,d_n-i}\nearrow x^*$ for any fixed $i\ge 1$; $\xi_{2n,d_{2n}}\nearrow y^*$ and $\xi_{2n-1,d_{2n-1}}\searrow y^*$ with $x^*<-c/b<y^*<x_{2,d_2}$.*

For these three cases, the respective best bounds for the set $\cup_{n\ge 1}R_n$ are $(-\infty,\,x^*)$, $(-\infty,\,y^*)$, and $(-\infty,\,r)$. Moreover, the sequence $\xi_{n,i}$ converges to $-\infty$ for any fixed $i\ge 1$.

When considering the root geometry problem of general recursive polynomial sequences of type $(0,1)$, it is acceptable to suppose that $\deg W_0(x)\le \deg W_1(x)$ and that the polynomial $W_0(x)$ is monic. Assume that $W_0(x)$ is a constant. Then we have $W_0(x)=1$.

Consider the polynomials

$$\widehat{W}_n(x) \;=\; (-1)^n W_n(-x). \tag{2.10}$$

It is routine to verify the recurrence

$$\widehat{W}_n(x) \;=\; -a\widehat{W}_{n-1}(x) + (-bx+c)\widehat{W}_{n-2}(x), \tag{2.11}$$

with $\widehat{W}_0(x)=1$ and $\widehat{W}_1(x)=x$. Then the roots of the polynomials $W_n(x)$ are the opposites of roots of the polynomials $\widehat{W}_n(x)$. From this point of view, when $a<0$, one may consider the root geometry of the polynomials $\widehat{W}_n(x)$, for which the coefficient $-a$ in Rec. (2.11) is positive. Therefore, we can suppose that $a>0$ from the beginning.

Provided that $a>0$. Then the case $b<0$ is unexplored. In fact, when $b<0$, both the degrees and the leading coefficients of the polynomials $W_n(x)$ may vary irregularly. We also note that dropping Condition (iii) may yield non-real-rooted polynomials $W_n(x)$. For example, when $a=1$, $b=-1$, and $c=-1$, we have $W_3(x)=-x^2-x-1$, which has no real roots. From the argument for supposing $a>0$, we do not handle the case $ab<0$ essentially.

We remark that in a general setting, beyond the genus polynomials of graphs, the polynomials $W_n(x)$ might have negative coefficients. In summary, this study of the root geometry of recursive polynomials of type $(0,1)$ has only two restrictions. One is that the polynomial $W_0(x)$ is a constant. The other is the assumption that the number b has the same sign as the number a.

2.2. **Some examples.** We now present several examples to illustrate our results.

Example 2.9. One kind of Fibonacci polynomials $W_n(x)$ is defined by the recursion

$$W_n(x) \;=\; W_{n-1}(x) + xW_{n-2}(x), \tag{2.12}$$

where $W_0(x) = 1$ and $W_1(x) = x + 1$; see [20, Table 3] and [27, A011973]. Accordingly,

$$a = b = 1, \qquad c = 0, \qquad \text{and} \qquad r = -1,$$

and we compute from Def. (2.9) that

$$x^* = -\frac{4c + a^2}{4b} = -\frac{1}{4} \qquad \text{and} \qquad r^* = x^* - \frac{a}{2} = -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4} > -1 = r.$$

By Theorem 2.6 (i), we know that each polynomial $W_n(x)$ is distinct-real-rooted and that all roots are less than $-1/4$. Also, for any $\epsilon > 0$, there exists a number $M' > 0$ such that every polynomial $W_n(x)$ with $n > M'$ has a root in the interval $(-1/4 - \epsilon,\, -1/4)$. Moreover, by the final conclusion of Theorem 2.6, we know that for any $N > 0$, there exists a number $M > 0$ such that every polynomial $W_n(x)$ with $n > M$ has a root less than $-N$.

In the next two examples, we examine how the set of convergent points is affected when we change the coefficient of $W_{n-2}(x)$ in Rec. (2.12) to $2x/5$ and to $x + 2$.

Example 2.10. Let $W_n(x)$ be the polynomial sequence defined by the recursion

$$W_n(x) \;=\; W_{n-1}(x) + \frac{2x}{5}W_{n-2}(x),$$

with initial values $W_0(x) = 1$ and $W_1(x) = x + 1$. We see that

$$a = 1, \qquad b = 2/5, \qquad c = 0, \qquad \text{and} \qquad r = -1.$$

We calculate from Def. (2.9) that

$$x^* = -\frac{4c + a^2}{4b} = -\frac{5}{8} < -\frac{3}{5} = y^* \qquad \text{and} \qquad r = -1 \in \left(-\frac{9}{8},\, 0\right) = \left(r^*,\, -\frac{c}{b}\right).$$

By Theorem 2.6, the polynomial $W_n(x)$ is distinct-real-rooted, and the largest roots converge to $-3/5$ increasingly. Moreover, for any $i \ge 1$, the roots $x_{n,\,d_n - i}$ converge to $-5/8$ increasingly as $n \to \infty$, and the roots $x_{n,\,i}$ converge to $-\infty$ decreasingly.

Example 2.11. Let $W_n(x)$ be the polynomial sequence defined by the recursion

$$W_n(x) \;=\; W_{n-1}(x) + (x + 2)W_{n-2}(x),$$

with initial values $W_0(x) = 1$ and $W_1(x) = x + 1$. Thus,

$$a = b = 1, \qquad c = 2, \qquad \text{and} \qquad r = -1.$$

We compute that $W_2(x) = 2x + 3$, and that

$$x^* = -\frac{9}{4}, \qquad r = -1 > -2 = -\frac{c}{b}, \qquad \text{and} \qquad y^* = -\sqrt{2}.$$

Therefore, we have $x^* < -c/b < x_{2,\,d_2}$. By Theorem 2.6, every polynomial $W_n(x)$ is distinct-real-rooted, and has exactly one root larger than $-9/4$. The largest roots converge to $-\sqrt{2}$ oscillatingly. Moreover, for any positive integer i, the roots $x_{n,\,d_n - i}$ converge to $-9/4$ increasingly as $n \to \infty$, and the roots $x_{n,\,i}$ converge to $-\infty$ decreasingly.

Example 2.12. This example illustrates how our results can be used to prove the real-rootedness of a sequence of partial genus polynomials. Let $D_n(x)$ be the polynomial sequence defined by the recursion

$$D_n(x) = 2D_{n-1}(x) + 8xD_{n-2}(x),$$

with $D_0(x) = 1$ and $D_1(x) = 2x$, which may be recognized by those familiar with enumerative research in topological graph theory (for example, see [8,10,14]) as a partial genus distribution for the closed-end ladder L_n, which is shown in Fig. 1.

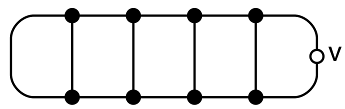

FIGURE 1. The closed-end ladder L_4 with a 2-valent root-vertex v.

The polynomial $D_n(x)$ is the generating function for the number of cellular imbeddings of the ladder L_n such that two different faces are incident on the root-vertex. By Theorem 2.6, each $D_n(x)$ is a distinct-real-rooted polynomial, and the root sequence $\xi_{n,\,d_n-i}$ converges to $-1/8$ for every nonnegative integer i. In particular, none of the polynomials $D_n(x)$ has a root larger than $-1/8$. Unfortunately, we do not yet know what topological information is implied by this convergent point.

3. Distinct Real-Rootedness

The proof of Theorem 2.8 begins here with an investigation of the real-rootedness of a $(0,1)$-sequence of polynomials. The remainder of the proof will be given in Sections 4 and 5.

For any polynomial $f(x)$, we follow the usual definition that

$$f(\pm\infty) = \lim_{x\to\pm\infty} f(x).$$

We start our analysis of $(0,1)$-sequences $\{W_n(x)\}_{n\geq 0}$ by finding a formula for the degree and the leading coefficient of each of the polynomials.

Lemma 3.1. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(0,1)$-sequence of polynomials, with t_n the leading coefficient of $W_n(x)$. Then*

$$d_n = \deg(W_n(x)) = \left\lfloor \frac{n+1}{2} \right\rfloor, \qquad t_{2n+1} = b^n, \qquad \text{and} \qquad t_{2n} = b^{n-1}(na+b).$$

Moreover, for all $n \geq 1$, we have

$$W_n(-\infty)(-1)^{d_n} \;=\; W_n(+\infty) \;=\; +\infty.$$

Proof. The formulas for the degree d_n and the leading coefficients t_n can be verified by induction on the integer n. For any polynomial $f(x)$ with positive leading coefficient, it is clear that

$$f(-\infty)(-1)^{\deg f(x)} = +\infty \qquad \text{and} \qquad f(+\infty) = +\infty.$$

Since $t_n > 0$, we infer that

$$W_n(-\infty)(-1)^{d_n} \;=\; W_n(+\infty) \;=\; +\infty.$$

The sign relations follow immediately. ∎

Using the intermediate value theorem for a $(0,1)$-sequence of polynomials, we derive the following criterion for their distinct-real-rootedness.

Theorem 3.2. *Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials, with zero-set R_n and degree d_n of the polynomial $W_n(x)$. Let $\beta \le -c/b$. Suppose that for some numbers $m, k \in \mathbb{N}$, we define*

$$T_m = R_m \cap (-\infty,\, \beta) \qquad \text{and} \qquad T_{m+1} = R_{m+1} \cap (-\infty,\, \beta), \tag{3.1}$$

and suppose, further, that

$$|T_m| \;=\; d_m - k, \tag{3.2}$$
$$|T_{m+1}| \;=\; d_{m+1} - k, \tag{3.3}$$
$$T_{m+1} \;\rtimes\; T_m, \qquad \text{and} \tag{3.4}$$
$$W_n(\beta)(-1)^k \;>\; 0, \tag{3.5}$$

for $n \in \{m,\, m+1,\, m+2\}$. Then there exists a set $T_{m+2} \subseteq R_{m+2} \cap (-\infty,\, \beta)$ such that

$$|T_{m+2}| \;=\; d_{m+2} - k \qquad \text{and} \qquad T_{m+2} \rtimes T_{m+1}.$$

Moreover, if

$$T_{m+2} \;=\; R_{m+2} \cap (-\infty,\, \beta), \tag{3.6}$$

then we have $T_{m+2} \bowtie T_m$.

Proof. By Eqs. (3.2) and (3.3) in the premises, we can suppose that

$$T_{m+1} = \{x_1,\, x_2,\, \ldots,\, x_p\} \qquad \text{and} \qquad T_m = \{y_1,\, y_2,\, \ldots,\, y_q\}$$

are ordered sets, where

$$p = d_{m+1} - k \qquad \text{and} \qquad q = d_m - k. \tag{3.7}$$

Def. (3.1) implies that $x_p < \beta$. In view of Relation (3.4), together with the premise $\beta \le -c/b$, we have the following ordering:

$$\cdots \;<\; y_{q-2} \;<\; x_{p-2} \;<\; y_{q-1} \;<\; x_{p-1} \;<\; y_q \;<\; x_p \;<\; \beta \;\le\; -c/b. \tag{3.8}$$

Note that Relation (3.4) also implies that $p \ge 1$ and $q \in \{p-1,\, p\}$. For convenience, let

$$x_0 = y_0 = -\infty \qquad \text{and} \qquad x_{p+1} = y_{q+1} = \beta.$$

We will apply Lemma 2.4 with

$$f(x) = W_{m+1}(x) \qquad \text{and} \qquad g(x) = W_m(x).$$

In this case, we have

$$X = R_{m+1} \qquad \text{and} \qquad Y = R_m.$$

Consequently, by Def. (3.1), we have

$$X' = R_{m+1} \cap (-\infty,\, \beta) = T_{m+1} \qquad \text{and} \qquad Y' = R_m \cap (-\infty,\, \beta) = T_m.$$

Then, Relation (3.4) reads $X' \rtimes Y'$. Taking $n = m$ in Ineq. (3.5) gives

$$W_m(\beta)(-1)^k \;>\; 0. \tag{3.9}$$

It follows that $W_m(\beta) \ne 0$. Therefore, we can use Ineq. (2.6), which gives that

$$W_m(x_i)W_m(\beta)(-1)^{p-i} \;>\; 0 \qquad \text{for all } i \in [p-q,\, p]. \tag{3.10}$$

Let $i \in [p]$. Since $x_i \in T_{m+1} \subseteq R_{m+1}$, we have

$$W_{m+1}(x_i) \;=\; 0.$$

Taking $n = m+2$ and $x = x_i$ in Rec. (2.7), we find

$$W_{m+2}(x_i) \;=\; (bx_i + c)W_m(x_i).$$

From Ineq. (3.8), we see that $x_i < -c/b$, and thus $bx_i + c \neq 0$. Therefore, we can substitute

$$W_m(x_i) \;=\; \frac{W_{m+2}(x_i)}{bx_i + c}$$

into Ineq. (3.10), which gives that

$$\frac{W_{m+2}(x_i)}{bx_i + c} W_m(\beta)(-1)^{p-i} \;>\; 0.$$

Since $b > 0$ and $x_i < -c/b$, we deduce that $bx_i + c < 0$. Thus the above inequality can be reduced to

$$W_{m+2}(x_i)W_m(\beta)(-1)^{p-i} \;<\; 0. \tag{3.11}$$

We notice that Ineq. (3.11) also holds true for $i = p+1$, namely,

$$W_{m+2}(x_{p+1})W_m(\beta)(-1)^{p-(p+1)} \;<\; 0,$$

that is,

$$W_{m+2}(\beta)W_m(\beta) \;>\; 0,$$

whose truth can be seen from Ineq. (3.5). Consequently, we can replace i by $(i+1)$ in Ineq. (3.11), which gives

$$W_{m+2}(x_{i+1})W_m(\beta)(-1)^{p-i-1} \;<\; 0.$$

Multiplying it by Ineq. (3.11), we obtain that

$$W_{m+2}(x_i)W_{m+2}(x_{i+1}) \;<\; 0.$$

By the intermediate value theorem, the polynomial $W_{m+2}(x)$ has a root in the interval $(x_i,\, x_{i+1})$. Let z_i be such a root.

When $i = 1$, Ineq. (3.11) is

$$W_{m+2}(x_1)W_m(\beta)(-1)^{p-1} \;<\; 0. \tag{3.12}$$

On the other hand, Lemma 3.1 gives

$$W_{m+2}(-\infty)(-1)^{d_{m+2}} \;>\; 0. \tag{3.13}$$

Multiplying Ineqs. (3.9), (3.12) and (3.13), we find

$$W_m(\beta)(-1)^k \cdot W_{m+2}(x_1)W_m(\beta)(-1)^{p-1} \cdot W_{m+2}(-\infty)(-1)^{d_{m+2}} \;<\; 0,$$

that is,

$$W_{m+2}(-\infty)W_{m+2}(x_1)(-1)^{d_{m+2}+k+p-1} \;<\; 0. \tag{3.14}$$

Recall from (3.7) that $p = d_{m+1} - k$, and from Lemma 3.1 that

$$d_{m+1} + d_{m+2} \;=\; m+2.$$

Inequality (3.14) implies that

$$W_{m+2}(-\infty)W_{m+2}(x_1)(-1)^m \;>\; 0.$$

Therefore, by the intermediate value theorem, the polynomial $W_{m+2}(x)$ has a root in the interval $(-\infty,\, x_1)$ when m is odd. Let z_0 be such a root. Define

$$T_{m+2} = \begin{cases} \{z_1,\, z_2,\, \ldots,\, z_p\}, & \text{if } m \text{ is even;} \\ \{z_1,\, z_2,\, \ldots,\, z_p\} \cup \{z_0\}, & \text{if } m \text{ is odd.} \end{cases} \tag{3.15}$$

We shall now show that this set T_{m+2} has the desired properties.

- For each $j \in [0, p]$, the number z_j is chosen to be a root of the polynomial $W_{m+2}(x)$. Therefore, $T_{m+2} \subseteq R_{m+2}$.
- For each $j \in [0, p]$, the number z_j is chosen from the interval $(x_j,\, x_{j+1})$, which is contained in the interval $(-\infty,\, \beta)$. Therefore, $T_{m+2} \subset (-\infty,\, \beta)$.
- From Def. (3.15), we see that
  - if m is even, then
  $$|T_{m+2}| \;=\; p \;=\; d_{m+1} - k \;=\; (m+2)/2 - k \;=\; d_{m+2} - k;$$
  - otherwise m is odd, then
  $$|T_{m+2}| \;=\; p + 1 \;=\; d_{m+1} - k + 1 \;=\; (m+3)/2 - k \;=\; d_{m+2} - k.$$

  Hence, in any case, we have that $|T_{m+2}| = d_{m+2} - k$.
- Since for all $j \in [0, p]$,
$$z_j \;\in\; (x_j,\, x_{j+1}), \tag{3.16}$$
we have $T_{m+2} \rtimes T_{m+1}$ according to Definition 2.3.

It remains to show that $T_{m+2} \bowtie T_m$. We apply Lemma 2.4 again, for

$$f(x) = W_{m+1}(x) \qquad \text{and} \qquad g(x) = W_m(x).$$

Taking $n = m + 1$ in Ineq. (3.5), we find

$$W_{m+1}(\beta)(-1)^k \;>\; 0. \tag{3.17}$$

It follows that $W_{m+1}(\beta) \neq 0$. Therefore, from Ineq. (2.5), we infer that

$$W_{m+1}(y_j)W_{m+1}(\beta)(-1)^{q-j} \;<\; 0 \tag{3.18}$$

for all $j \in [q + 1 - p,\, q + 1]$. Now, let $j \in [q]$. Taking $n = m + 2$ and $x = y_j$ in Rec. (1.1) gives

$$W_{m+2}(y_j) \;=\; aW_{m+1}(y_j).$$

Since $a > 0$, we can substitute

$$W_{m+1}(y_j) \;=\; \frac{W_{m+2}(y_j)}{a}$$

into Ineq. (3.18), and obtain that

$$W_{m+2}(y_j)W_{m+1}(\beta)(-1)^{q-j} \;<\; 0. \tag{3.19}$$

It is noticeable from Ineq. (3.5) that Ineq. (3.19) holds also for $j = q + 1$. Therefore, we can replace j by $(j + 1)$ in Ineq. (3.19), which gives

$$W_{m+2}(y_{j+1})W_{m+1}(\beta)(-1)^{q-j-1} \;<\; 0.$$

Multiplying it with Ineq. (3.19), we find

$$W_{m+2}(y_j)W_{m+2}(y_{j+1}) \;<\; 0.$$

By the intermediate value theorem, the polynomial $W_{m+2}(x)$ has an odd number of roots in the interval $(y_j,\, y_{j+1})$ for each $j \in [q]$. If some of the intervals contains at least three roots, then $W_{m+2}(x)$ has at least $(q+2)$ roots in the interval union

$$\cup_{j=1}^{q}(y_j,\, y_{j+1}) \subset (-\infty,\, \beta).$$

On the other hand, the premise Eq. (3.6) and the result

$$|T_{m+2}| \;=\; d_{m+2} - k \;=\; d_m + 1 - k \;=\; q + 1$$

imply that the polynomial $W_{m+2}(x)$ has exactly $(q+1)$ roots in the interval $(-\infty,\, \beta)$. This contradiction yields that $W_{m+2}(x)$ has exactly one root in each of the intervals $(y_j,\, y_{j+1})$. Since

$$z_p \;\in\; (x_p,\, \beta) \;\subset\; (y_q,\, \beta) \;=\; (y_q,\, y_{q+1}),$$

we infer that for each $j \in [p]$,

$$z_j \in (y_{q-p+j},\, y_{q-p+j+1}). \tag{3.20}$$

If m is even, then $q = p - 1$, and Relation (3.20) can be written as

$$z_1 \;<\; y_1 \;<\; z_2 \;<\; y_2 \;<\; \cdots \;<\; z_{p-1} \;<\; y_q \;<\; z_p \;<\; \beta. \tag{3.21}$$

Recall that $T_{m+2} = \{z_1,\, z_2,\, \ldots,\, z_p\}$ in this case, and $T_m = \{y_1,\, y_2,\, \cdots,\, y_q\}$. Inequality (3.21) implies immediately $T_{m+2} \bowtie T_m$.

Otherwise m is odd, then $q = p$, and Relation (3.20) reads

$$y_1 \;<\; z_1 \;<\; y_2 \;<\; z_2 \;<\; \cdots \;<\; y_p \;<\; z_p \;<\; \beta.$$

Recall that $T_{m+2} = \{z_0,\, z_1,\, \ldots,\, z_p\}$ in this case. It remains to show $z_0 < y_1$. In fact, when $j = 1$, Ineq. (3.18) becomes

$$W_{m+1}(y_1)W_{m+1}(\beta)(-1)^{q-1} \;<\; 0. \tag{3.22}$$

Multiplying Ineqs. (3.13), (3.17) and (3.22), we obtain

$$W_{m+2}(-\infty)(-1)^{d_{m+2}} \cdot W_{m+1}(\beta)(-1)^{k} \cdot W_{m+1}(y_1)W_{m+1}(\beta)(-1)^{q-1} \;<\; 0,$$

that is,

$$W_{m+2}(-\infty)W_{m+1}(y_1) \;<\; 0.$$

Thus by the intermediate value theorem, the polynomial $W_{m+2}(x)$ has an odd number of roots less than y_1. From Eq. (3.6) and Relation (3.16), we infer that the number z_0 is the unique root of $W_{m+2}(x)$ which is smaller than x_1. Since m is odd, we have $y_1 < x_1$, and thus z_0 must be the unique root of $W_{m+2}(x)$ which is smaller than y_1. This completes the proof. □

The usage of the above interlacing method dates back at least to Harper [15], who established the real-rootedness of the Bell polynomials in this way.

Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials, with the zero-set R_n of the polynomial $W_n(x)$. It is direct to compute that $W_2(x) = (a+b)x + c$. Therefore, the polynomials $W_1(x)$ and $W_2(x)$ are real-rooted, and

$$R_1 = \{0\} \qquad \text{and} \qquad R_2 = \{-c/(a+b)\}. \tag{3.23}$$

In the remainder of this section, we will use Theorem 3.2 frequently. We will always set the constant k to be either 0 or 1. The following lemma is for the particular case $k = 0$.

Lemma 3.3. *Let $\{W_n(x)\}_{n\ge0}$ be a $(0,1)$-sequence of polynomials, with the zero-set R_n of the polynomial $W_n(x)$. Let $c < 0$, and*

$$-c/(a+b) < \beta \le -c/b. \tag{3.24}$$

Let N be a positive integer. If $W_m(\beta) > 0$ for all $m \in [N]$, then we have

$$\begin{aligned} R_m &\subset (-\infty,\, \beta) && \text{for all } m \in [N+2], \\ R_{m+1} &\rtimes R_m && \text{for all } m \in [N+1], \qquad \text{and} \\ R_{m+2} &\bowtie R_m && \text{for all } m \in [N]. \end{aligned}$$

In particular, if $W_m(\beta) > 0$ for all $m \ge 1$, then the above three relations hold for all $m \ge 1$.

Proof. First, we show the following relations by induction on the integer m:

$$R_m \subset (-\infty,\, \beta), \qquad R_{m+1} \subset (-\infty,\, \beta), \qquad \text{and} \qquad R_{m+1} \rtimes R_m \qquad \text{for all } m \in [N+1]. \tag{3.25}$$

Recall from the equations in (3.23) that $R_1 = \{0\}$ and $R_2 = \{-c/(a+b)\}$. When $m = 1$, the relations in (3.25) reduce to

$$R_1 \subset (-\infty,\, \beta), \qquad R_2 \subset (-\infty,\, \beta), \qquad \text{and} \qquad R_2 \rtimes R_1,$$

that is,

$$0 < \beta, \qquad -c/(a+b) < \beta, \qquad \text{and} \qquad 0 < -c/(a+b).$$

Since $a, b > 0$, the above relations hold by the negativity of the number c and Ineq. (3.24) in the premises. Suppose that the relations in (3.25) hold for $m \in [N]$, and we need to show them for $m+1$.

Let $k = 0$. The upper bound $-c/b$ of the parameter β is as same as that in Theorem 3.2. We are going to verify the conditions: Eqs. (3.2) and (3.3), Relation (3.4), and Ineq. (3.5).

- From Def. (3.1) and the induction hypothesis $R_m \subset (-\infty,\, \beta)$, we infer that $T_m = R_m \cap (-\infty,\, \beta) = R_m$. It follows that $|T_m| = |R_m| = d_m$, i.e., Eq. (3.2) holds.
- Similarly, we have $T_{m+1} = R_{m+1}$, i.e., Eq. (3.3) holds.
- Thus, by the induction hypothesis we have that $R_{m+1} \rtimes R_m$, which is equivalent to Relation (3.4).
- Since $k = 0$, the premise $W_m(\beta) > 0$ for all $m \ge 1$ justifies Ineq. (3.5).

Therefore, we can apply Theorem 3.2 and obtain the existence of a set $T_{m+2} \subseteq R_{m+2} \cap (-\infty,\, \beta)$ such that $T_{m+2} \rtimes T_{m+1}$ and $|T_{m+2}| = d_{m+2}$. Since the sets T_{m+2} and R_{m+2} have the same cardinality d_{m+2}, we obtain that

$$T_{m+2} = R_{m+2}. \tag{3.26}$$

Consequently, the result $T_{m+2} \subset (-\infty,\, \beta)$ becomes the desired relation

$$R_{m+2} \subset (-\infty,\, \beta); \tag{3.27}$$

and the result $T_{m+2} \rtimes T_{m+1}$ becomes the desired relation $R_{m+2} \rtimes R_{m+1}$. This completes the induction proof for the relations in (3.25).

By Eq. (3.26) and Relation (3.27), we infer that $T_{m+2} = R_{m+2} \cap (-\infty,\, \beta)$, which is exactly Eq. (3.6). Hence, by Theorem 3.2, we derive that $T_{m+2} \bowtie T_m$, i.e., $R_{m+2} \bowtie R_m$, which completes the proof. $\square$

In order to continue with our discussions, we fix more parameters of $(0,1)$-sequences of polynomials. Inspired by Lemma 2.1, we introduce the following notations.

Definition 3.4. Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials. We define

$$\begin{aligned}\Delta(x) &= a^2+4(bx+c) = 4bx+a^2+4c,\\ g^{\pm}(x) &= \big(2x-a\pm\sqrt{\Delta(x)}\,\big)/2 = \big(2x-a\pm\sqrt{4bx+a^2+4c}\big)/2,\\ g(x) &= g^-(x)g^+(x) = x^2-(a+b)x-c.\end{aligned}$$

We denote the zeros of the functions $B(x)=bx+c$, $\Delta(x)$ and $g^+(x)$ by

$$x_B = -\frac{c}{b}, \qquad x_\Delta = -\frac{a^2+4c}{4b}, \qquad \text{and} \qquad x_g = \frac{(a+b)-\sqrt{(a+b)^2+4c}}{2}, \tag{3.28}$$

respectively. We also define

$$n_0=\frac{2ab}{a^2+2ab+4c}.$$

We observe that Lemma 2.1 implies the following:

$$W_n(x_B) = a^{n-1}W_1(x_B), \tag{3.29}$$

$$W_n(x_\Delta) = \left(1+\frac{n(2x_\Delta-a)}{a}\right)\left(\frac{a}{2}\right)^n, \text{ and} \tag{3.30}$$

$$W_n(x_g) = x_g^n. \tag{3.31}$$

The following technical lemma provides the ordering among the numbers x_Δ, x_g, x_B, and 0, for the sake of determining the sign of the value $W_n(x)$ for specific numbers x in the proofs of Theorem 3.8.

Lemma 3.5. *Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials. Then we have the following.*

(i) *If $\frac{4c}{a^2+2ab}\le -1$, then $W_n(x_\Delta)>0$ for all $n\ge 1$.*

(ii) *If $\frac{4c}{a^2+2ab}> -1$,then $x_g\in\mathbb{R}$, $x_\Delta<x_g$, and*

$$W_n(x_\Delta)\ge 0 \iff n\le n_0=\frac{2ab}{a^2+2ab+4c}, \tag{3.32}$$

where the equality on the left hand side of Relation (3.32) holds if and only if the equality on the right hand side holds. Moreover, if $c>0$, then $W_n(x_\Delta)<0$ and $x_B<x_g<0$; otherwise, we have $0<x_g<x_B$.

Proof. See Appendix B. □

Below is an example illustrating the cases $-(a^2+2ab)/4<c<0$ and $c>0$, respectively.

Example 3.6. Let $\{W_n(x)\}_{n\ge 1}$ be a $(0,1)$-sequence of polynomials defined by the recursion

$$W_n(x)=W_{n-1}(x)+(x-1/2)W_{n-2}(x),$$

with initial values $W_0(x)=1$ and $W_1(x)=x$. We see that $a=b=1$ and $c=-1/2$. It is direct to compute that

$$\frac{4c}{a^2+2ab}=-\frac{2}{3}, \qquad n_0=2, \qquad x_B=\frac{1}{2}, \qquad x_\Delta=\frac{1}{4}, \qquad \text{and} \qquad x_g=1-\frac{1}{\sqrt{2}}.$$

By Relation (3.32), we infer that $W_n(1/4) \ge 0$ if and only if $n \le 2$, and that $0 < x_g < x_B$.

Example 3.7. Let $\{W_n(x)\}_{n\ge 1}$ be a $(0,1)$-sequence of polynomials with parameters specified as in Definition 3.4, and with $a = b = 1$ and $c = 1$, which puts us in Case (i); here we have

$$W_n(x) \;=\; W_{n-1}(x) + (x+1)W_{n-2}(x),$$

with $W_0(x) = 1$ and $W_1(x) = x$. This time, we have $c > 0$. By Definition 3.4, we have $x_\Delta = -5/4$, $x_g = 1-\sqrt{2}$, $x_B = -1$, and $n_0 = 2/7$. These data correspond to the inequalities $x_B < x_g$ and $x_\Delta < x_g$ in the conclusion of Lemma 3.5.

We are now ready to establish the real-rootedness of every polynomial $W_n(x)$.

Theorem 3.8. *Let $\{W_n(x)\}_{n\ge 1}$ be a $(0,1)$-sequence of polynomials. Then every polynomial $W_n(x)$ is distinct-real-rooted. Moreover, let us denote the ordered zero-set of $W_n(x)$ by R_n, and let $y_n = \max R_n$ be the largest real root of the polynomial $W_n(x)$. For all $n \ge 1$, we may conclude the following:*

(i) *if $c < 0$, then $y_n < x_B$, $R_{n+1} \rtimes R_n$, and $R_{n+2} \bowtie R_n$.*
(ii) *if $c > 0$, then $y_n > x_B$, $R'_{n+1} \subset (-\infty, x_\Delta)$, $R'_{n+2} \rtimes R'_{n+1}$, and $R'_{n+2} \bowtie R'_n$, where $R'_n = R_n \setminus \{y_n\}$.*

Proof. From Eq. (3.29), we see that

$$cW_n(x_B) < 0. \tag{3.33}$$

Below we will show (i) and (ii) individually.

(i) Let $c < 0$. Then Ineq. (3.33) reduces to $W_n(x_B) > 0$ for all $n \ge 1$. Take $\beta = -c/b$. Then Ineq. (3.24) holds trivially. By Lemma 3.3, we deduce that $R_n \subset (-\infty, x_B)$, $R_{n+1} \rtimes R_n$ and $R_{n+2} \bowtie R_n$ for all $n \ge 1$.

(ii) Let $c > 0$. Then Ineq. (3.33) implies that $W_n(x_B) < 0$. By Lemma 3.1, we have $W_n(+\infty) > 0$. Therefore, by the intermediate value theorem, the polynomial $W_n(x)$ has a real root in this interval $(x_B, +\infty)$. In particular, the largest root y_n is larger than x_B. Note that $x_\Delta = -(a^2+4c)/(4b) < -c/b = x_B$. Thus, we have

$$x_\Delta < x_B < y_n \qquad \text{for all } n \ge 1. \tag{3.34}$$

For the remaining desired relations, it suffices to show the following:

$$R'_n \subset (-\infty, x_\Delta), \quad R'_{n+1} \subset (-\infty, x_\Delta), \quad R'_{n+1} \rtimes R'_n, \quad R'_{n+1} \bowtie R'_{n-1}, \qquad \text{for all } n \ge 2. \tag{3.35}$$

We proceed by induction on n. Consider $n = 2$. Since $d_1 = d_2 = 1$, we have $R'_1 = R'_2 = \emptyset$. Since $a, b, c > 0$, from Def. (3.28), we have

$$x_\Delta = -(a^2+4c)/(4b) < 0.$$

In view of Eq. (3.30), we deduce that

$$W_n(x_\Delta) \;=\; \left(1 + \frac{n(2x_\Delta - a)}{a}\right)\left(\frac{a}{2}\right)^n < 0, \qquad \text{for all } n \ge 1. \tag{3.36}$$

In particular, we have $W_3(x_\Delta) < 0$. On the other hand, Lemma 3.1 gives that $W_3(-\infty)(-1)^{d_3} > 0$. Since $d_3 = 2$, it reduces to $W_3(-\infty) > 0$. Therefore, by the intermediate value theorem, we infer that the polynomial $W_3(x)$ has a root, say, r_3, in the interval $(-\infty, x_\Delta)$. From Ineq. (3.34), we see that

$y_3 > x_\Delta$, and thus, $r_3 < x_\Delta < y_3$. It follows that $R_3 = \{r_3,\, y_3\}$, and thus, $R_3' = \{r_3\}$. Therefore, the relations in (3.35) for $n = 2$ are respectively

$$\emptyset \subset (-\infty,\, x_\Delta), \qquad \{r_3\} \subset (-\infty,\, x_\Delta), \qquad \{r_3\} \rtimes \emptyset, \qquad \{r_3\} \bowtie \emptyset,$$

all of which hold trivially, except the second one holds since $r_3 < x_\Delta$.

Suppose that all the relations in (3.35) hold for some $n \ge 2$, by induction, it suffices to show that

$$R_{n+2}' \subset (-\infty,\, x_\Delta), \qquad R_{n+2}' \rtimes R_{n+1}', \qquad \text{and} \qquad R_{n+2}' \bowtie R_n'. \tag{3.37}$$

In applying Theorem 3.2, we set

$$k = 1, \qquad \beta = x_\Delta, \qquad \text{and} \qquad m = n.$$

We shall verify the conditions: Eqs. (3.2) and (3.3), Relation (3.4), and Ineq. (3.5).

- From Def. (3.1), we have $T_n = R_n \cap (-\infty,\, x_\Delta)$. Note that in the zero-set R_n, except the largest root y_n, which is not in the interval $(-\infty,\, x_\Delta)$ by Ineq. (3.34), all the other roots (whose union is the set R_n') are in the interval $(-\infty,\, x_\Delta)$ by the relations (3.35). Therefore, we infer that $R_n \cap (-\infty,\, x_\Delta) = R_n'$, and thus, $T_n = R_n'$. It follows that $|T_n| = |R_n'| = d_n - 1$, which verifies Eq. (3.2).
- Similarly, we have $T_{n+1} = R_{n+1}'$, and Eq. (3.3) holds true.
- Consequently, the hypothesis $T_{n+1} \rtimes T_n$ in (3.35) can be rewritten as $R_{n+1}' \rtimes R_n'$, which verifies Relation (3.4).
- Inequality (3.36) with $k = 1$ guarantees the truth of Ineq. (3.5).

By Theorem 3.2, there exists a set $T_{n+2} \subseteq R_{n+2} \cap (-\infty,\, x_\Delta)$ such that $|T_{n+2}| = d_{n+2} - 1$ and $T_{n+2} \rtimes T_{n+1}$. From Ineq. (3.34), we see that $y_{n+2} > x_\Delta$. It follows that

$$R_{n+2} \cap (-\infty,\, x_\Delta) = (R_{n+2}' \cup \{y_{n+2}\}) \cap (-\infty,\, x_\Delta) \subseteq R_{n+2}'. \tag{3.38}$$

Thus, we have $T_{n+2} \subseteq R_{n+2}'$. Since the sets T_{n+2} and R_{n+2}' have the same cardinality $d_{n+2} - 1$, we infer that $T_{n+2} = R_{n+2}'$. Now, the result $T_{n+2} \subset (-\infty,\, x_\Delta)$ is one of the desired relations:

$$R_{n+2}' \subset (-\infty,\, x_\Delta); \tag{3.39}$$

the result $T_{n+2} \rtimes T_{n+1}$ is another one of the desired relations:

$$R_{n+2}' \rtimes R_{n+1}'.$$

In view of our goal (3.37), it suffices to show that $R_{n+2}' \bowtie R_n'$, i.e., $T_{n+2} \bowtie T_n$. By Theorem 3.2, it suffices to verify Eq. (3.6), i.e.,

$$R_{n+2}' = R_{n+2} \cap (-\infty,\, x_\Delta).$$

In view of Relation (3.39), we deduce that $R_{n+2}' \subseteq R_{n+2} \cap (-\infty,\, x_\Delta)$. Together with Relation (3.38), we find the above equation, which completes the proof. □

Continuing Examples 3.6 and 3.7, we present the approximate values of roots in the ordered set $R_n = \{\xi_{n,1},\, \xi_{n,2},\, \ldots,\, \xi_{n,d_n}\}$.

Example 3.9. This example continues Example 3.6. Table 1 illustrates that for $n \le 8$, we have

$$y_n = \max R_n \;<\; x_B \;=\; 1/2, \qquad R_{n+1} \;\rtimes\; R_n \qquad \text{and} \qquad R_{n+2} \;\bowtie\; R_n.$$

A more careful observation suggests that the second largest root $\xi_{n,\,d_n-1}$ is bounded by the number $x_\Delta = 0.25$. In fact, this is true in general, which motivates Theorem 4.1.

TABLE 1. The approximate roots of $W_n(x)$ $(1 \le n \le 8)$ in Example 3.6.

	$\xi_{n,\,d_n-3}$	$\xi_{n,\,d_n-2}$	$\xi_{n,\,d_n-1}$	$\xi_{n,\,d_n} = y_n$
$n = 1$				0
$n = 2$				0.2500
$n = 3$			−1.7807	0.2807
$n = 4$			−0.2886	0.2886
$n = 5$		−4.2912	0	0.2912
$n = 6$		−1.0218	0.1046	0.2922
$n = 7$	−7.5833	−0.3639	0.1547	0.2926
$n = 8$	−1.9561	−0.1194	0.1827	0.2927

Example 3.10. This example continues Example 3.7. Table 2 illustrates that for $n \le 8$,

$$\xi_{n,\,d_n-1} < x_\Delta = -5/4 \qquad \text{and} \qquad y_n > x_B = -1.$$

A more careful observation suggests that the largest root y_n converges to the point x_g in an oscillating manner, where x_g equals approximately -0.4142. In fact, this convergence is true in general; see Theorems 4.1 and 5.5.

TABLE 2. The approximate roots of $W_n(x)$ $(1 \le n \le 8)$ in Example 3.7.

	$\xi_{n,\,d_n-3}$	$\xi_{n,\,d_n-2}$	$\xi_{n,\,d_n-1}$	$\xi_{n,\,d_n} = y_n$
$n = 1$				0
$n = 2$				−0.5000
$n = 3$			−2.6180	−0.3819
$n = 4$			−1.5773	−0.4226
$n = 5$		−5.1819	−1.4064	−0.4116
$n = 6$		−2.2405	−1.3444	−0.4149
$n = 7$	−8.5525	−1.7194	−1.3140	−0.4139
$n = 8$	−3.1548	−1.5342	−1.2966	−0.4142

4. BOUND ON THE ZERO-SET R_n

As consequence of the real-rootedness of the $(0, 1)$-sequence polynomials $\{W_n(x)\}_{n\ge0}$, we improve the bound of the zero-set R_n of $W_n(x)$.

Theorem 4.1. *Let $\{W_n(x)\}_{n\ge0}$ be a $(0, 1)$-sequence of polynomials, with zero-set R_n of the polynomial $W_n(x)$. Let $y_n = \max R_n$ be the largest real root of $W_n(x)$, and let $R'_n = R_n \setminus \{y_n\}$. Then we have the following.*

(i) *If $c \le -(a^2 + 2ab)/4$, then $R_n \subset (-\infty, x_\Delta)$ for all $n \ge 1$.*

(ii) *If $-(a^2 + 2ab)/4 < c < 0$, then we have*
   - *$R_n \subset (-\infty, x_\Delta)$, for $n < n_0$;*

– $R_n' \subset (-\infty, x_\Delta)$ *and* $y_n = x_\Delta$, *for* $n = n_0$;
– $R_n' \subset (-\infty, x_\Delta)$ *and* $y_n \in (x_\Delta,\, x_g)$, *for* $n > n_0$;

(iii) *If* $c > 0$, *then we have* $R_n' \subset (-\infty, x_\Delta)$, *and*

$$x_B < y_2 < y_4 < y_6 < \cdots < y_{2n} < \cdots < x_g < \cdots < y_{2n-1} < \cdots < y_5 < y_3 < y_1 = 0. \tag{4.1}$$

Proof. We treat the three cases individually.
(i) Let $c \le -(a^2 + 2ab)/4$. Since $a, b > 0$, it is routine to check that

$$-c/(a+b) \;<\; -(a^2+4c)/(4b) \;<\; -c/b,$$

which verifies Ineq. (3.24) for $\beta = x_\Delta$. By Lemma 3.5, we have

$$W_n(x_\Delta) > 0 \qquad \text{for all } n \ge 1. \tag{4.2}$$

Now, by Lemma 3.3, we deduce that $R_n \subset (-\infty,\, x_\Delta)$ for all $n \ge 1$.

(ii) Let $-(a^2 + 2ab)/4 < c < 0$.

Case $n < n_0$**.** Recall from the equations in (3.23) that $R_1 = \{0\}$ and $R_2 = \{-c/(a+b)\}$. If $n_0 \le 1$, then nothing needs to be shown in this case. Next suppose that $n_0 > 1$, i.e., $a^2 + 4c < 0$. Together with $b > 0$, this implies that

$$0 < -(a^2+4c)/(4b) = x_\Delta,$$

i.e., $R_1 \subset (-\infty,\, x_\Delta)$. If $n_0 \le 2$, then nothing else needs to be shown. And then suppose that $n_0 > 2$, i.e., $a^2 + ab + 4c < 0$. Together with $a, b > 0$, it is routine to check that

$$-\,c/(a+b) < -(a^2+4c)/(4b), \tag{4.3}$$

i.e., $R_2 \subset (-\infty,\, x_\Delta)$. If $n_0 \le 3$, nothing else needs to be shown. So we may suppose that $n_0 > 3$.

Let $N = \lceil n_0 \rceil - 3$. Since $n_0 > 3$, the integer N is positive. Take $\beta = x_\Delta$. From $x_\Delta < -c/b$, together with Ineq. (4.3), we see that Ineq. (3.24) holds true. By Lemma 3.5, we have $W_n(x_\Delta) > 0$ for all $n \in [N]$. By Lemma 3.3, we have $R_n \subset (-\infty,\, x_\Delta)$ for all $n \in [N+2] = [\lceil n_0 \rceil - 1]$, i.e., for all $n < n_0$.

Case $n = n_0$. It follows that the number n_0 is an integer. By Lemma 3.5, we have $W_{n_0}(x_\Delta) = 0$. It suffices to show that the polynomial $W_{n_0}(x)$ has no roots larger than the number x_Δ. If $n_0 = 1$, then the polynomial $W_{n_0}(x) = W_1(x) = x$ has only one root. So we are done. Suppose that $n_0 \ge 2$. By the interlacing property $R_{n_0} \rtimes R_{n_0-1}$ obtained in Theorem 3.8, we see that the second largest root of the polynomial $W_{n_0}(x)$ is less than the largest root of the polynomial $W_{n_0-1}(x)$, which is less than the number x_Δ, in view of the case $n < n_0$. This completes the proof for the case $n = n_0$.

Case $n > n_0$**.** First, we show that $y_n < x_g$, i.e., $R_n \subset (-\infty,\, x_g)$. We do this by applying Lemma 2.4 for $\beta = x_g$. Recall from Def. (3.28) that $x_g = (a + b - \sqrt{(a+b)^2 + 4c})/2$. Since $a, b > 0$ and $-(a^2 + 2ab)/4 < c < 0$, it is routine to check that

$$-\,c/(a+b) \;<\; (a+b-\sqrt{(a+b)^2+4c})/2. \tag{4.4}$$

By Lemma 3.5 (ii), we have

$$\max(0,\, x_\Delta) < x_g < x_B. \tag{4.5}$$

The particular inequality $x_g < x_B$, together with Ineq. (4.4), verifies Ineq. (3.24). On the other hand, since $x_g > 0$, Eq. (3.31) implies that

$$W_n(x_g) \;>\; 0, \qquad \text{for all } n \ge 1. \tag{4.6}$$

From Lemma 3.3, we deduce that $R_n \subset (-\infty,\, x_g)$ for all $n \ge 1$.

By Lemma 3.5, we have $W_n(x_\Delta) < 0$. In view of Ineq. (4.6), the polynomial $W_n(x)$ has different signs at the ends of the interval $(x_\Delta,\, x_g)$. Therefore, the polynomial $W_n(x)$ has an odd number, say p_n, of roots in the interval $(x_\Delta,\, x_g)$. In particular, we have

$$p_n \ge 1 \qquad \text{for all } n > n_0. \tag{4.7}$$

It suffices to show that $p_n = 1$, for all $n > n_0$. We proceed the proof by induction on n. Note that the largest root of the polynomial $W_{\lfloor n_0 \rfloor}(x)$ is less than or equal to the number x_Δ. By the interlacing property $R_{\lfloor n_0 \rfloor+1} \rtimes R_{\lfloor n_0 \rfloor}$, the polynomial $W_{\lfloor n_0 \rfloor+1}(x)$ has at most one root larger than the number x_Δ, i.e., $p_{\lfloor n_0 \rfloor+1} \le 1$. In view of Ineq. (4.7), we deduce that $p_{\lfloor n_0 \rfloor+1} = 1$. Thus, we can suppose that there is some $n > n_0$ such that $p_k = 1$ for all $n_0 < k \le n$. If $n \le 2$, then the degree $d_n \le 1$. It follows immediately that $p_n = 1$. Suppose that $n \ge 3$. By the interlacing property $R_{n+1} \rtimes R_n$, the third largest root of the polynomial $W_{n+1}(x)$ is less than the second largest root of the polynomial $W_n(x)$, which is at most x_Δ since $p_n = 1$. Therefore, the polynomial $W_{n+1}(x)$ has at most two roots larger than the number x_Δ, i.e., $p_n \le 2$. Since the integer p_n is odd, in view of Ineq. (4.7), we infer that $p_n = 1$. This completes and the induction and hence the proof of (ii).

(iii) Let $c > 0$. The bound for the set R_n' has been confirmed in Theorem 3.8. It suffices to show Ineq. (4.1). By Theorem 3.8, we have $y_n > x_B$ for all $n \ge 1$. It suffices to show that

$$y_{2n} \;<\; y_{2n+2} \;<\; x_g \qquad \text{and} \tag{4.8}$$

$$x_g \;<\; y_{2n+1} \;<\; y_{2n-1} \tag{4.9}$$

for all $n \ge 0$, where $y_0 = x_B$ and $y_{-1} = +\infty$. We proceed by induction on the integer n. When $n = 0$, the desired Ineqs. (4.8) and (4.9) become $y_2 < x_g < y_1$, i.e.,

$$-c/(a+b) < (a+b-\sqrt{(a+b)^2+4c})/2 < 0.$$

Since $a, b, c > 0$, it is routine to check the truth of the above inequalities. Now, based on the induction hypothesis that

$$y_{2n} \;<\; x_g \;<\; y_{2n-1}, \tag{4.10}$$

we are going to show Ineqs. (4.8) and (4.9).

Since the number y_{2n} is largest real root of the polynomial $W_{2n}(x)$, and $y_{2n-1} > y_{2n}$ by Ineq. (4.10), we infer that the value $W_{2n}(y_{2n-1})$ has the same sign as the limit $W_{2n}(+\infty)$, which is positive by Lemma 3.1. Therefore, we find $W_{2n}(y_{2n-1}) > 0$. Replacing n by $2n-1$ in Rec. (2.7), and taking $x = y_{2n-1}$, we obtain that

$$W_{2n+1}(y_{2n-1}) \;=\; aW_{2n}(y_{2n-1}) \;>\; 0. \tag{4.11}$$

On the other hand, by Lemma 3.5, we have $x_g < 0$. Thus from Eq. (3.31), we infer that

$$W_n(x_g)(-1)^n \;>\; 0, \qquad \text{for all } n \ge 1. \tag{4.12}$$

In particular, we have $W_{2n+1}(x_g) < 0$. Together with Ineq. (4.11), we see that the polynomial $W_{2n+1}(x)$ attains different signs at the ends of the interval $(x_g,\, y_{2n-1})$. By the intermediate value theorem, the polynomial $W_{2n+1}(x)$ has a root in the interval $(x_g,\, y_{2n-1})$. By Theorem 3.8, only the largest root y_{2n+1} of the polynomial $W_{2n+1}(x)$ is larger than the number x_B. Since $x_B < x_g$, we conclude that $y_{2n+1} \in (x_g,\, y_{2n-1})$. This proves Ineq. (4.9).

Denote by z_{2n+1} the second largest root of the polynomial $W_{2n+1}(x)$. From the interlacing property $R_{2n+1} \rtimes R_{2n}$, we infer that

$$W_{2n+1}(x)W_{2n+1}(+\infty) < 0 \qquad \text{for all } x \in (z_{2n+1},\, y_{2n+1}).$$

By Lemma 3.1, we see that the limit $W_{2n+1}(+\infty) = +\infty$. It follows that

$$W_{2n+1}(x) < 0 \qquad \text{for all } x \in (z_{2n+1},\, y_{2n+1}). \tag{4.13}$$

Now, from Ineqs. (4.9) and (4.10), we see that $y_{2n} < x_g < y_{2n+1}$. From Theorem 3.8, we see that $z_{2n+1} < x_B < y_{2n}$. By Eq. (4.13), we infer that $W_{2n+1}(y_{2n}) < 0$.

Replacing n by $2n+2$ in Rec. (2.7), and taking $x = y_{2n}$, we obtain that

$$W_{2n+2}(y_{2n}) \;=\; aW_{2n+1}(y_{2n}) \;<\; 0. \tag{4.14}$$

By Ineq. (4.12), we have $W_{2n+2}(x_g) > 0$. By the intermediate value theorem, the polynomial $W_{2n+2}(x)$ has a root in the interval $(y_{2n},\, x_g)$. Since only its largest root is larger than the number x_B, and since $y_{2n} > x_B$, we conclude that $y_{2n+2} \in (y_{2n},\, x_g)$. This proves Ineq. (4.8), which completed the induction. □

In summary, we see that "almost all" roots lie in the open interval $(-\infty, x_\Delta)$. Precisely speaking, when $c \le -(a^2+2ab)/4$, all roots lie in $(-\infty, x_\Delta)$; when $c > -(a^2+2ab)/4$, only the largest root of the polynomial $W_n(x)$ is possibly but "eventually" larger than x_Δ, with maximum value $\max(x_g,\, 0)$.

Before ending this section, we mention that the recurrence system defined by Rec. (2.7) can be solved always by transforming the polynomials $W_n(x)$ into Chebyshev polynomials. More precisely, by induction and by the fact that Chebyshev polynomials of the second kind satisfy the recursion

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

with initial conditions $U_0(t) = 1$ and $U_1(t) = t$, we obtain that

$$W_n(x) \;=\; \sqrt{-bx-c}^{\,n}\left(\frac{x}{\sqrt{-bx-c}}U_{n-1}\left(\frac{a}{2\sqrt{-bx-c}}\right) - U_{n-2}\left(\frac{a}{2\sqrt{-bx-c}}\right)\right).$$

By this, it is now clear that all roots of $W_n(x)$ are real and bounded.

5. Limit Points of the Zero-Set R_n

In this section, we show that one of the intervals $(-\infty, x_\Delta)$, $(-\infty,\, x_g)$, and $(-\infty,\, y_2)$ is the best bound of all roots, depending on the range of the constant term c of the linear polynomial coefficient $B(x) = bx + c$. More precisely, we will demonstrate three limit points of the zero-set $\cup_{n\ge1}R_n$ over the course of several subsections. We say that a proposition *holds for large* n, if there exists a number N such that the proposition holds whenever $n > N$.

5.1. **The number x_g can be a limit point.** The following lemma will help determine all limit points of the zero-set $\cup_{n\geq 1} R_n$, which are larger than the number x_Δ.

Lemma 5.1. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(0,1)$-sequence of polynomials, with zero-set R_n of the polynomial $W_n(x)$. Let $x_0 \neq x_g$ and $\Delta(x_0) > 0$. Then $(x_0 - x_g)W_n(x_0) > 0$, for large n.*

Proof. See Appendix C. □

Using Lemma 5.1, we can confirm that the roots outside the interval $(-\infty, x_\Delta)$ converges to the number x_g when $n \to \infty$ as follows.

Theorem 5.2. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(0,1)$-sequence of polynomials, with zero-set R_n of the polynomial $W_n(x)$. Let $y_n = \max R_n$ be the largest real root of $W_n(x)$.*

(i) *If $-(a^2 + 2ab)/4 < c < 0$, then we have $y_n \nearrow x_g$.*
(ii) *If $c > 0$, then we have $y_{2n} \nearrow x_g$ and $y_{2n+1} \searrow x_g$.*

Proof. We treat the two cases individually.

(i) Suppose that $-(a^2 + 2ab)/4 < c < 0$. Since $R_{n+1} \rtimes R_n$, the sequence y_n increases. In virtue of Theorem 4.1, we have $y_n < x_g$ for all $n \geq 1$. Therefore, the sequence y_n converges to a finite number y^* as $n \to \infty$. If $y^* < x_g$, then there exists $x_0 \in (x_\Delta, x_g)$ such that the values $W_n(x_0)$ and $W_n(x_g)$ have the same sign for large n, i.e., $W_n(x_0) > 0$ for large n; see Ineq. (4.6). This contradicts Lemma 5.1. Hence, we have that $y_n \nearrow x_g$.

(ii) Suppose that $c > 0$. From Theorem 4.1, we see that the sequence y_{2n} converges to a finite number y^*. Then we have $x_B < y^* \leq x_g$. Suppose to the contrary that $y^* < x_g$, so there exists $x_0 \in (y^*, x_g)$ such that the numbers $W_{2n}(x_0)$ and $W_{2n}(x_g)$ have the same sign for large n, i.e., $W_{2n}(x_0) > 0$ for large n; see Ineq. (4.12). This contradicts Lemma 5.1. Along the same line, we can show the convergence $y_{2n+1} \searrow x_g$, which completes the proof. □

An illustration for the convergences above can be found in Tables 1 and 2.

5.2. **The number x_Δ is a limit point.** In an analog with Lemma 5.1, we give a characterization of the sign of the value $W_n(x_0)$ for the case $\Delta(x_0) < 0$. This time the criterion for the sign is for all positive integers n. We define l_{x_0} to be the straight line $\sqrt{-\Delta(x_0)}\,x + (2x_0 - a)y = 0$, and the radian $\theta(x_0)$ to be $\arctan \frac{\sqrt{-\Delta(x_0)}}{a}$.

Lemma 5.3. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(0,1)$-sequence of polynomials, and let $\Delta(x_0) < 0$.*

- *If the radian $n\theta(x_0)$ lies to the left of the line l_{x_0}, then $W_n(x_0) < 0$;*
- *If the radian $n\theta(x_0)$ lies on the line l_{x_0}, then $W_n(x_0) = 0$;*
- *If the radian $n\theta(x_0)$ lies to the right of the line l_{x_0}, then $W_n(x_0) > 0$.*

Proof. See Appendix D. □

Let us get some illustration of this characterization from the example below.

Example 5.4. This example continues Example 3.6. Take $x_0 = -1$, we have

$$\Delta(x_0) \;=\; -5 \;<\; 0 \qquad \text{and} \qquad \theta(x_0) \;=\; \arctan(\sqrt{5}).$$

The line l_{x_0} becomes $\sqrt{5}x - 3y = 0$. Thus a radian ϕ lies to the left of the line l_{x_0} if and only if

$$\phi \in \big(\arctan(\sqrt{5}/3) + 2\ell\pi,\ \arctan(\sqrt{5}/3) + (2\ell+1)\pi\big) \quad \text{for some integer } \ell. \tag{5.1}$$

By approximating $\arctan(\sqrt{5}/3) \approx 0.6405$, we have that

$$\theta(x_0) \approx 1.1502, \qquad 2\theta(x_0) \approx 2.3005, \qquad \text{and} \qquad 3\theta(x_0) \approx 3.4507.$$

By Relation (5.1), we deduce that θ_{x_0}, $2\theta_{x_0}$ and $3\theta_{x_0}$ lie to the left of the line l_{x_0}. In the same way we can deduce that the radians

$$4\theta(x_0) \approx 4.6010, \qquad 5\theta(x_0) \approx 5.7513, \qquad \text{and} \qquad 6\theta(x_0) \approx 6.9015$$

lie to the right of the line l_{x_0}. The truth is, as one may compute directly, that

$$\begin{aligned} W_1(-1) &= -1, & W_2(-1) &= -5/2, & W_3(-1) &= -1,\\ W_4(-1) &= 11/4, & W_5(-1) &= 17/4, & W_6(-1) &= 1/8. \end{aligned}$$

The above data verifies the fact that $W_n(x_0) < 0$ for $n \in \{1,2,3\}$, and that $W_n(x_0) > 0$ for $n \in \{4,5,6\}$, coinciding with the characterization.

Now we are ready to justify that the number x_Δ is a limit point.

Theorem 5.5. *Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials, with ordered zero-set*

$$R_n \;=\; \{\xi_{n,1},\, \xi_{n,2}\, \ldots,\, \xi_{n,d_n}\}$$

of the polynomial $W_n(x)$. Then we have

$$\lim_{n\to\infty} \xi_{n,\,d_n - i} \;=\; x_\Delta \tag{5.2}$$

for all $i \ge 0$ if $c \le -(a^2+2ab)/4$; and for all $i \ge 1$ otherwise.

Proof. Let $c \le -(a^2+2ab)/4$. We will show Eq. (5.2) for all $i \ge 0$. As will be seen, the other case can be done in the same vein.

From the interlacing property obtained in Theorem 3.8, we see that the sequence $\{\xi_{n,\,d_n-i}\}_{n\ge1}$ increases and all its members are less than the number x_Δ, which implies that it converges to a number which is at most x_Δ. Suppose, by way of contradiction, that the limit point of the sequence $\{\xi_{n,\,d_n-i}\}_{n\ge1}$ is not the point x_Δ.

When $i = 0$, there exists a point $x_0 < x_\Delta$ such that the numbers $W_n(x_0)$ and $W_n(x_\Delta)$ have the same sign, i.e., we have $W_n(x_0) > 0$ for large n. Therefore, by Lemma 5.3, the radian $n\theta(x_0)$ resides in certain one side of the line l_{x_0} forever for large n. This is impossible because $\theta(x_0) < \pi/2$. Hence we deduce that $\lim_{n\to\infty} \xi_{n,d_n} \;=\; x_\Delta$.

Now for $i = 1$, the sequence $\{\xi_{n,\,d_n-1}\}_{n\ge1}$ converges to some point less than x_Δ. Thus, there exists a number $x_1 < x_\Delta$ such that the numbers $W_n(x_1)$ and $W_n(x_\Delta)$ have distinct signs, i.e., we have $W_n(x_1) < 0$ for large n. Here again, the radian $\theta(x_1)$ resides in certain one side of the line l_{x_1} for large n, a contradiction. This confirms the truth of Eq. (5.2) for $i = 1$. Continuing in this way, we can deduce that for a general $i \ge 2$, there exists a number $x_i < x_\Delta$, such that

$$W_n(x_i)(-1)^i \;>\; 0 \qquad \text{for large } n,$$

which contradicts Lemma 5.3. Hence, we conclude that Eq. (5.2) holds true for all $i \geq 0$.

Now we consider the other possibility that $c > -(a^2 + 2ab)/4$. In fact, the above contradiction idea still works. This is because that, whatever sign does the value $W_n(x_\Delta)$ have, it is a fixed sign. However, the sign of the value $W_n(x_0)$ for any point $x_0 < x_\Delta$ can not be invariant for large n. This completes the proof. □

5.3. **The negative infinity is a limit point.** We are ready to study the negative infinity as a limit point.

Theorem 5.6. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(0,1)$-sequence of polynomials, with ordered zero-set*

$$R_n \;=\; \{\xi_{n,1},\, \xi_{n,2},\, \ldots,\, \xi_{n,d_n}\}$$

of the polynomial $W_n(x)$. Then we have

$$\lim_{n\to\infty} \xi_{n,i} \;=\; -\infty, \qquad \textit{for all } i \geq 1.$$

Proof. From the interlacing property $R_{n+2} \bowtie R_n$ obtained in Theorem 3.8, we see that the sequences $\{\xi_{2n,\,i}\}_{n\geq 1}$ decreases, and so does the sequence $\{\xi_{2n-1,\,i}\}_{n\geq 1}$. Therefore, these two sequences converge respectively. We shall show that both of these sequences converge to the negative infinity.

Suppose, by way of contradiction, that the sequence $\{\xi_{2n,\,1}\}_{n\geq 1}$ converges to some real number x^*. Then for any number $x_0 < x^*$, the number $W_n(x_0)$ has the sign of $W_n(-\infty)$. It follows that the sign of the number $W_n(x_0)$ would not change for large n, which contradicts Lemma 5.3. This proves that $\lim_{n\to\infty} \xi_{2n,i} \;=\; -\infty$ for $i = 1$. Its truth for general i, in fact, along the same lines, if it does not hold for some $i \geq 2$, then we can deduce the existence of a number x_i such that $x_i < x_\Delta$ and that the sign of the number $W_n(x_i)$ keeps invariant for large n, which leads to a contradiction.

Along the same lines, we can prove that $\lim_{n\to\infty} \xi_{2n-1,\,i} \;=\; -\infty$, for all $i \geq 1$.

Now, for any fixed $i \geq 1$, the subsequences $\{\xi_{2n,\,i}\}_{n\geq 1}$ and $\{\xi_{2n-1,\,i}\}_{n\geq 1}$ converge to the same point $-\infty$. Hence, the joint sequence $\{\xi_{n,i}\}_{n\geq 1}$ converges to the negative infinity as well, which completes the proof. □

For an illustration for the convergences in Theorems 5.5 and 5.6, the reader can refer to Tables 1 and 2.

References

[1] F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, *Mem. Amer. Math. Soc.* **413** (1989).

[2] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, *Contemp. Math.* **178** (1994), 71–89.

[3] A.L. Cauchy, Exercises de mathematique, *Oeuvres* **2**(9) (1829), 122.

[4] Y. Chen and Y. Liu, On a conjecture of S. Stahl, *Canad. J. Math.* **62** (2010), 1058–1059.

[5] Y. Chen, T. Mansour, and Q. Zou, Embedding distributions and Chebyshev polynomials, *Graphs Combin.* **28** (2012), 597–614.

[6] S. Fisk, Polynomials, roots, and interlacing, arXiv: math/0612833.

[7] M. Fujiwara, Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung, *Tôhoku Math. J.* **10** (1916), 167–171.

[8] M. Furst, J.L. Gross, and R. Statman, Genus distributions for two class of graphs, *J. Combin. Theory Ser. B* **46** (1989), 523–534.

[9] J.L. Gross, Genus distributions of cubic outerplanar graphs, *J. Graph Algorithms Appl.* **15** (2011), 295–316.
[10] J.L. Gross and M.L. Furst, Hierarchy for imbedding-distribution invariants of a graph, *J. Graph Theory* **11** (1987), 205–220.
[11] J.L. Gross, I.F. Khan, and M.I. Poshni, Genus distributions for iterated claws, *Electron. J. Combin.* **21**(1) (2014), #P1,12.
[12] J.L. Gross, T. Mansour, T.W. Tucker, and D.G.L. Wang, Iterated claws have real-rooted genus polynomials, Keynote address at GEMS13 Conference in Slovakia, manuscript. Conference proceedings to appear in *Ars Math. Contemporanea*.
[13] J.L. Gross, T. Mansour, T.W. Tucker, and D.G.L. Wang, Log-concavity of combinations of sequences and applications to genus distributions, *SIAM J. Discrete Math.* **29**(2) (2015), 1002–1029.
[14] J.L. Gross, D.P. Robbins, and T.W. Tucker, Genus distributions for bouquets of circles, *J. Combin. Theory Ser. B* **47** (1989), 292–306.
[15] L.H. Harper, Stirling behavior is asymptotically normal, *Ann. Math. Statist.* **38**(2) (1967), 410–414.
[16] H.P. Hirst and W.T. Macey, Bounding the roots of polynomials, *Coll. Math. J.* **28**(4) (1997), 292.
[17] V.L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Mathematics and Its Applications, Volume 256, Kluwer Academic Publishers, Dordrecht/Boston/London, Springer, 1993.
[18] J.B. Lasserre, Polynomials with all zeros real and in a prescribed interval, *J. Alg. Combin.* **16** (2002), 231–237.
[19] M. Marden, *Geometry of Polynomials*, Issue 3 of Mathematical Surveys, 2nd ed., Amer. Math. Soc., 1985.
[20] D. Merlini, R. Sprugnoli and M.C. Verri, The tennis ball problem, *J. Combin. Theory Ser. A* **99** (2002), 307–344.
[21] N. Obreshkov, *Zeros of Polynomials*, Volume 7 of Bulgarian Academic Monographs, Marin Drinov Academic Publishing House, 2003.
[22] J. Pitman, Probabilistic bounds on the coefficients of polynomials with only real zeros, *J. Combin. Theory Ser. A* **77** (1997), 279–303.
[23] M.I. Poshni, I.F. Khan, and J.L. Gross, Genus distributions of iterated 3-wheels and 3-prisms, *Ars Math. Contemporanea* **7** (2014), 423–440.
[24] V.V. Prasolov, *Polynomials*, Volume 11 of Algorithms and Computation in Mathematics, Springer-Verlag Berlin Heidelberg, 2010.
[25] Q.I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, London Math. Soc. Monogr. New Ser. 26., The Clarendon Press, Oxford University Press, Oxford, 2002.
[26] G.-C. Rota and D. Sharp, Mathematics, Philosophy, and Artificial Intelligence, *Los Alamos Sci.*, **12**, 1985.
[27] N.J.A. Sloane, The on-line encyclopedia of integer sequences, `http://oeis.org`.
[28] S. Stahl, On the zeros of some genus polynomials, *Canad. J. Math.* **49** (1997), 617–640.

Appendix A. Proof of Lemma 2.4

Let $x_0 = y_0 = -\infty$ and $y_{q+1} = \beta$. The interlacing property $X' \rtimes Y'$ in the premises implies that $p \ge 1$ and $q \in \{p-1, p\}$. Since $X' \subset (-\infty, \beta)$, we infer that $x_p < \beta$. Therefore, we have that

$$\cdots < y_{q-2} < x_{p-2} < y_{q-1} < x_{p-1} < y_q < x_p < \beta.$$

We shall show Ineqs. (2.5) and (2.6) respectively.

Let $i \in [p]$. From the definition $X' = X \cap (-\infty, \beta)$ and the interlacing property $X' \rtimes Y'$ in the premises, we see that the number x_{p+1-i} is the unique root of the polynomial $f(x)$ in the interval (y_{q+1-i}, y_{q+2-i}). Suppose that $f(\beta) \neq 0$. By the intermediate value theorem, we infer that

$$f(y_{q+1-i})f(y_{q+2-i}) < 0,$$

that is,

$$\begin{aligned} f(y_q)f(\beta) &< 0 \qquad (i=1),\\ f(y_{q-1})f(y_q) &< 0 \qquad (i=2),\\ &\vdots\\ f(y_{q-p+1})f(y_{q-p+2}) &< 0 \qquad (i=p). \end{aligned}$$

Multiplying the first i inequalities in the above list results in that

$$f(y_{q+1-i})f(\beta)(-1)^{i-1} < 0.$$

Replacing i by $q+1-j$ in it yields Ineq. (2.5) for $j \in [q+1-p,\, q]$. When $j = q+1$, since $y_{q+1} = \beta$ stands as a premise, Ineq. (2.5) holds true trivially.

From the definition $Y' = Y \cap (-\infty, \beta)$, we deduce that the polynomial $g(x)$ has no roots in the interval (y_q, β). Suppose that $g(\beta) \neq 0$. By the intermediate value theorem, we infer that $g(x)g(\beta) > 0$ for all $x \in (y_q, \beta)$. In particular, we have

$$g(x_p)g(\beta) > 0, \tag{A.1}$$

which is Ineq. (2.6) for $j = p$. Below we can suppose that $p \geq 2$, and thus, $q \geq 1$.

Let $j \in [q]$. Similar to the previous proof, we have

$$g(x_{p-j})g(x_{p+1-j}) < 0,$$

that is,

$$\begin{aligned} g(x_{p-1})g(x_p) &< 0 \qquad (j=1),\\ g(x_{p-2})g(x_{p-1}) &< 0 \qquad (j=2),\\ &\vdots\\ g(x_{p-q})g(x_{p-q+1}) &< 0 \qquad (j=q). \end{aligned}$$

Multiplying the first j inequalities in the above list, we find that

$$g(x_{p-j})g(x_p)(-1)^{j-1} < 0.$$

Multiplying it by Ineq. (A.1) results in that

$$g(x_{p-j})g(\beta)(-1)^{j-1} < 0.$$

Replacing j by $p-i$ in it yields that

$$g(x_i)g(\beta)(-1)^{p-i} > 0.$$

Together with Ineq. (A.1), we obtain Ineq. (2.6). This completes the proof. $\square$

Appendix B. Proof of Lemma 3.5

From Eq. (3.30), we have that

$$W_n(x_\Delta)\big(a + n(2x_\Delta - a)\big) \;>\; 0. \tag{B.1}$$

(i) If $c \le -(a^2+2ab)/4$, then we have

$$x_\Delta \;=\; -\frac{a^2+4c}{4b} \;\ge\; -\frac{a^2-(a^2+2ab)}{4b} \;=\; \frac{a}{2}, \tag{B.2}$$

that is, $2x_\Delta - a \ge 0$. It follows that

$$a + n(2x_\Delta - a) \;>\; 0 \qquad \text{for all } n \ge 1.$$

By Ineq. (B.1), we obtain that $W_n(x_\Delta) > 0$.

(ii) Below we suppose that $c > -(a^2+2ab)/4$. From the Ineq. (B.2), we see that $2x_\Delta - a < 0$. If $n < n_0$, then we have

$$a + n(2x_\Delta - a) \;>\; a + \frac{2ab}{a^2+2ab+4c}\cdot\left(2\left(-\frac{a^2+4c}{4b}\right) - a\right) \;=\; 0,$$

which, by Ineq. (B.1), implies that $W_n(x_\Delta) > 0$. Similarly, if $n = n_0$ then we have that

$$a + n(2x_\Delta - a) = 0,$$

and thus $W_n(x_\Delta) = 0$ by Ineq. (B.1); and if $n > n_0$ then we have that $a + n(2x_\Delta - a) < 0$, and thus $W_n(x_\Delta) < 0$ by Ineq. (B.1).

When $c > 0$, we have that

$$n_0 \;=\; \frac{2ab}{a^2+2ab+4c} \;<\; 1.$$

Therefore, the case $n > n_0$ happens for all $n \ge 1$, that is, $W_n(x_\Delta) < 0$. Thus, by Definition 3.4 we obtain that $x_g < 0$. Moreover, we have

$$x_g - x_B \;=\; \frac{(a+b) - \sqrt{(a+b)^2+4c}}{2} + \frac{c}{b} \;=\; \frac{ab+b^2+2c - b\sqrt{a^2+2ab+b^2+4c}}{2b}.$$

Thus, to show that $x_g > x_B$, it suffices to show that

$$(ab+b^2+2c)^2 \;>\; b^2(a^2+2ab+b^2+4c).$$

By direct calculation, this inequality is equivalent to $4c(ab+c) > 0$, which is true since $a, b, c > 0$.

When $c < 0$, we have $x_g > 0$ from Definition 3.4 straightforwardly. Suppose to the contrary that $x_g \ge x_B$. It follows that

$$2c + b(a+b) \;\ge\; b\sqrt{(a+b)^2+4c} \;>\; 0.$$

Solving the inequality $(2c+b(a+b))^2 \ge (b\sqrt{(a+b)^2+4c})^2$ with $c < 0$, we see that $c \le -ab$. On the one hand, by solving $2c + b(a+b) > 0$, we get

$$-b(a+b)/2 < c \le -ab,$$

which implies that $a < b$. On the other hand, we have

$$-(a^2+2ab)/4 < c \le -ab,$$

which implies that $a > 2b$. Hence, we obtain $2b < a < b$, a contradiction. This proves $x_g < x_B$ when $-(a^2+2ab)/4 < c < 0$. □

Appendix C. Proof of Lemma 5.1

By Lemma 2.1, the value $W_n(x_0)$ can be recast as the following form

$$W_n(x_0) \;=\; \frac{\big(A(x_0)+\sqrt{\Delta(x_0)}\,\big)^n}{2^n\sqrt{\Delta(x_0)}}\left[g^+(x_0)-g^-(x_0)\left(\frac{A(x_0)-\sqrt{\Delta(x_0)}}{A(x_0)+\sqrt{\Delta(x_0)}}\right)^n\right].$$

Since $A(x_0)=a>0$ and $\sqrt{\Delta(x_0)}>0$, we deduce that

$$\left|\frac{A(x_0)-\sqrt{\Delta(x_0)}}{A(x_0)+\sqrt{\Delta(x_0)}}\right| \;<\; 1.$$

Thus we obtain that

$$W_n(x_0)g^+(x_0)>0 \qquad \text{for large } n. \tag{C.1}$$

Note that the function

$$2g^+(x) \;=\; 2x-a+\sqrt{4(bx+c)+a^2}$$

is increasing. Since $g^+(x_g)=0$, we infer that

$$(x_0-x_g)g^+(x_0)>0.$$

In view of Ineq. (C.1), we conclude that

$$(x_0-x_g)W_n(x_0)>0$$

for large n, which completes the proof. □

Appendix D. Proof of Lemma 5.3

By Lemma 2.1, the sign of the value $W_n(x_0)$ is equal to the sign of the value $F=\cos\theta+\ell\sin\theta$, where $\theta=n\theta(x_0)$, and $\ell=(2x_0-a)/\sqrt{-\Delta(x_0)}$.

If $x_0=a/2$, then the line l_{x_0} becomes the imagine axis $x=0$. In this case, the sign of the value $W_n(x_0)$ is determined by the sign of the value $\cos\theta$. In other words, we have $W_n(x_0)>0$ if and only if the radian $n\theta_0$ lies in the right open half-plane, and $W_n(x_0)<0$ if and only if the radian $n\theta_0$ lies in the left open half-plane.

Below we can suppose that $x_0\neq a/2$. It follows that $\ell\neq 0$.

- Assume that $\ell>0$. It is elementary to find the following equivalence relation

$$F>0 \quad\Longleftrightarrow\quad \begin{cases} \tan\theta>-1/\ell, & \text{if } \cos\theta>0;\\ \sin\theta>0, & \text{if } \cos\theta=0;\\ \tan\theta<-1/\ell, & \text{if } \cos\theta<0. \end{cases}$$

  In this case, we have $F>0$ if and only if the radian θ lies to the right of the line $y=-x/\ell$, that is, of the line l_{x_0}. By symmetry, we have $F<0$ if and only if the radian θ lies to the left of the line l_{x_0}. It follows immediately that $F=0$ if and only if the radian θ lies on the line l_{x_0}.

- Now suppose that $\ell < 0$. Then we have the following equivalence relation in the same vein:

$$F > 0 \quad \Longleftrightarrow \quad \begin{cases} \tan\theta < -1/\ell, & \text{if } \cos\theta > 0; \\ \sin\theta < 0, & \text{if } \cos\theta = 0; \\ \tan\theta > -1/\ell, & \text{if } \cos\theta < 0. \end{cases}$$

  In this case, we have the same desired characterization.

This completes the proof. □

Department of Computer Science, Columbia University, New York, NY 10027, USA

E-mail address: gross@cs.columbia.edu

Department of Mathematics, University of Haifa, 31905 Haifa, Israel

E-mail address: tmansour@univ.haifa.ac.il

Department of Mathematics, Colgate University, Hamilton, NY 13346, USA

E-mail address: ttucker@colgate.edu

†School of Mathematics and Statistics, Beijing Institute of Technology, 102488 Beijing, P. R. China, ‡Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, 102488 Beijing, P. R. China

E-mail address: glw@bit.edu.cn