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Genus distribution of graph amalgamations: Pasting when one root has arbitrary degree

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Abstract

This paper concerns counting the imbeddings of a graph in a surface. In the first installment of our current work, we showed how to calculate the genus distribution of an iterated amalgamation of copies of a graph whose genus distribution is already known and is further analyzed into a *partitioned genus distribution* (which is defined for a *double-rooted graph*). Our methods were restricted there to the case with two 2-valent roots. In this sequel we substantially extend the method in order to allow one of the two roots to have arbitrarily high valence.

Keywords: Graph, genus distribution, vertex-amalgamation.

Math. Subj. Class.: 05C10

1 Introduction

We continue the development of methods for counting imbeddings of interesting families of graphs in a range of surfaces. We are primarily concerned here with deriving recursions, rather than with exact formulas. It may be helpful to precede the reading of this paper with a light reading of [13].

By the *vertex-amalgamation* of the rooted graphs (G, t) and (H, u), we mean the graph obtained from their disjoint union by merging the roots t and u. We denote the operation of amalgamation by an asterisk, i.e.,

$$(G, t) * (H, u) \;=\; (X, w)$$

where X is the amalgamated graph and w the merged root.

Terminology. We take a *graph* to be connected and an *imbedding* to be cellular and orientable, unless it is evident from context that something else is intended. A graph need

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not be simple, i.e., it may have self-loops and multiple edges between two vertices. We use the words *degree* and *valence* of a vertex to mean the same thing. Each edge has two *edge-ends*, in the topological sense, even if it has only one endpoint.

Abbreviation. We abbreviate face-boundary walk as *fb-walk*.

Notation. The *degree* of a vertex y is denoted $deg(y)$. The *genus* of a surface S is denoted $\gamma(S)$. The *number of imbeddings* of a graph G in the surface S_i of genus i is denoted g_i. The sequence $\{g_i(G) \mid i \geq 0\}$ is called the *genus distribution* of the graph G. The terminology generally follows [15] and [1]. For additional background, see [3], [24], or [34].

Prior work concerned with the number of imbeddings of a graph in a minimum-genus surface includes [2], [8], [9], and [19]. Prior work concerned with counting imbeddings in all orientable surfaces or in all surfaces includes [4], [5], [7], [12], [14], [20], [21], [22], [23], [25], [27], [28], [29], [30], [31], [32], and [33].

Remark 1.1. Some of the calculations in this paper are quite intricate, and it appears that taking the direct approach here to amalgamating two graphs at roots of arbitrarily high degree might be formidable. We observe that a vertex of arbitrary degree can be split (by inverse contraction) into two vertices of smaller degree. Effects on the genus distribution that arise from splitting a vertex are described by [11].

Imbeddings induced by an amalgamation of two imbedded graphs

We say that the pair of imbeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ *induce* the set of imbeddings of $X = G * H$ whose rotations have the same cyclic orderings as in G and H, and that this set of imbeddings of X is the result of *amalgamating the two imbeddings* $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$.

Proposition 1.2. *For any two imbeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ of graphs into surfaces, the cardinality of the set of imbeddings of the amalgamated graph $(X, w) = (G, t) * (H, u)$ whose rotation systems are consistent with the imbeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ is*

$$(deg(u) + deg(t) - 1) \cdot \binom{deg(t) + deg(u) - 2}{deg(u) - 1} \tag{1.1}$$

Proof. Formula (1.1) is a symmetrization of Formula (1.1) of [13]. □

In the amalgamation $(G, t)*(H, u) = (X, w)$, when one of the roots t and u is 1-valent, the genus distribution of the resulting graph is easily derivable via bar-amalgamations (see [12]). For the case where

$$deg(t) = deg(u) = 2,$$

methods for calculating the genus distribution are developed in [13]. For the purposes of this paper, we assume that $deg(t) = 2$ and $deg(u) = n \geq 2$. A pair of such imbeddings $\iota_G : G \to S_G$ and $\iota_H : H \to S_H$ induce, in accordance with Formula (1.1), $n^2 + n$ imbeddings of the amalgamated graph X. We observe that for each such imbedding $\iota_X : X \to S_X$, we have

$$\gamma(S_X) = \begin{cases} \gamma(S_G) + \gamma(S_H) & \text{or} \\ \gamma(S_G) + \gamma(S_H) + 1 \end{cases}$$

Terminology. The difference $\gamma(S_X) - (\gamma(S_G) + \gamma(S_H))$ is called the *genus increment of the amalgamation*, or more briefly, the *genus increment* or *increment*.

Proposition 1.3. *In any vertex-amalgamation $(G, t) * (H, u) = (X, w)$ of two graphs, the increment of genus lies within these bounds:*

$$\left\lceil \frac{1 - deg(t) - deg(u)}{2} \right\rceil \leq \gamma(S_X) - (\gamma(S_G) + \gamma(S_H)) \leq \left\lfloor \frac{deg(t) + deg(u) - 2}{2} \right\rfloor$$

Proof. See [13]. □

Double-rooted graphs

By a *double-rooted graph* (H, u, v) we mean a graph with two vertices designated as roots. Double-rooted graphs were first introduced in [13] as they lend themselves natrually to iterated amalgmation. For the purposes of this paper, root u is assumed to have degree $n \geq 2$, whereas root v is 2-valent. Our focus here, is the graph amalgamation $(G, t) * (H, u, v)$ when $deg(t) = deg(v) = 2$ and $deg(u) = n \geq 2$. This is illustrated in Figure 1.

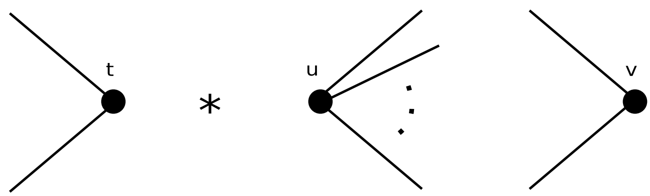

Figure 1: $(G, t) * (H, u, v)$ when $deg(t) = deg(v) = 2$ and $deg(u) = n \geq 2$.

When two single-rooted graphs are amalgamated, the amalgamated graph has the merged vertices of amalgamation as its root. If we iteratively amalgamate several single-rooted graphs, we obtain a graph with a root of whose degree is the sum of the degrees of the constituent roots. We use double-rooted graphs when we want to calculate the genus distribution of a chain of copies (as in §3 and §4) of the same graph (or of different graphs).

2 Double-root partials and productions

The genus distribution of the set of imbeddings of $(X, w) = (G, t) * (H, u)$ whose rotation systems are consistent with those of fixed imbeddings $G \to S_G$ and $H \to S_H$, depends only on $\gamma(S_G)$, $\gamma(S_H)$, and the respective numbers of faces of the imbeddings $G \to S_G$ and $H \to S_h$ in which the two vertices of amalgamation t and u lie. Accordingly, we partition the imbeddings of a single-rooted graph (G, t) with $deg(t) = 2$ in a surface of genus i into the subset of *type-d_i imbeddings*, in which root t lies on two distinct fb-walks, and the subset of *type-s_i imbeddings*, in which root t occurs twice on the same fb-walk. Moreover, we define

$$\begin{aligned} d_i(G, t) &= \text{the number of imbeddings of type-}d_i\text{, and} \\ s_i(G, t) &= \text{the number of imbeddings of type-}s_i. \end{aligned}$$

Thus,

$$g_i(G,t) \;=\; d_i(G,t) + s_i(G,t).$$

The numbers $d_i(G,t)$ and $s_i(G,t)$ are called *single-root partials*. The sequences $\{d_i(G,t) \mid i \geq 0\}$ and $\{s_i(G,t) \mid i \geq 0\}$ are called *partial genus distributions*.

Since $deg(u) = n$ in a double-rooted graph (H,u,v), there are n face corners incident at u (i.e., u occurs n times in the fb-walks — we will call them *u-corners* from now on), some or all of which might belong to the same face.

Suppose further that the n occurrences of root u in fb-walks of different faces are distributed according to the partition $p_1p_2\cdots p_r$ of n (where r is the number of faces incident at root u). For each such partition $p_1p_2\cdots p_r$, we define the following *double-root partials* of the genus distribution of a graph (H,u,v), such that root u is n-valent and root v is 2-valent:

$f_{p_1p_2\cdots p_r}d_i$ = the number of imbeddings of H such that the n occurrences of root u are distributed according to the partition $p_1p_2\cdots p_r$, and the 2 occurrences of v lie on two *different* fb-walks.

$f_{p_1p_2\cdots p_r}s_i$ = the number of imbeddings of H such that the n occurrences of root u are distributed according to the partition $p_1p_2\cdots p_r$, and the 2 occurrences of v lie on the *same* fb-walk.

Notation. We write the partition $p_1p_2\cdots p_r$ of an integer in non-ascending order.

A *production* for an amalgamation

$$(G,t)*(H,u,v) \;=\; (X,v)$$

of a single-rooted graph (G,t) with a double-rooted graph (H,u,v) (where $deg(t) = deg(v) = 2$, and $deg(u) \geq 2$) is an expression of the form

$$\begin{aligned} p_i(G,t)*q_j(H,u,v) \quad \longrightarrow \quad & \alpha_1\, d_{i+j}(G*H,v) \;+\; \alpha_2\, d_{i+j+1}(G*H,v) \\ & +\alpha_3\, s_{i+j}(G*H,v) \;+\; \alpha_4\, s_{i+j+1}(G*H,v) \end{aligned}$$

where p_i is a single-root partial and q_j is a double-root partial, and where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are integers. It means that amalgamation of a type-p_i imbedding of graph (G,u) and a type-q_j imbedding of graph (H,u,v) induces a set of α_1 type-d_{i+j}, α_2 type-d_{i+j+1}, α_3 type-s_{i+j}, and α_4 type-s_{i+j+1} imbeddings of $G*H$. We often write such a rule in the form

$$p_i * q_j \;\longrightarrow\; \alpha_1\, d_{i+j} \;+\; \alpha_2\, d_{i+j+1} + \alpha_3\, s_{i+j} \;+\; \alpha_4\, s_{i+j+1}$$

Sub-partials of $f_{p_1p_2\cdots p_r}d_i$

In the course of developing productions for amalgamating a single-rooted graph (G,t) to a double-rooted graph (H,u,v), we shall discover that we sometimes need to refine a double-root partial into sub-partials. The following two types of numbers are the *sub-partials of* $f_{p_1p_2\cdots p_r}d_i$:

$f_{p_1p_2\cdots p_r}d'_i$	=	the number of type-$f_{p_1p_2\cdots p_r}d_i$ imbeddings such that at most one of the r fb-walks incident at u is the same as one of the two fb-walks incident at v;
$f_{p_1p_2\cdots p_r}d_i^{(p_l,p_m)}$	=	the number of type-$f_{p_1p_2\cdots p_r}d_i$ imbeddings such that the two fb-walks (corresponding to subscripts l and m) incident at v have p_l and p_m occurrences of u, where $l < m$ (so that, in general, $p_l \geq p_m$), and $r > 1$.

Note that the value of the latter sub-partial of a graph (H, u, v) would be the same for any two pairs (p_a, p_b) and (p_l, p_m) such that $(p_l, p_m) = (p_a, p_b)$. Also note that, in general, we have

$$f_{p_1p_2\cdots p_r}d_i = f_{p_1p_2\cdots p_r}d'_i + \sum_{\substack{\text{over all distinct} \\ \text{pairs } (p_l,p_m) \\ \text{with } l<m}} f_{p_1p_2\cdots p_r}d_i^{(p_l,p_m)}$$

Example 2.1. For instance, $f_{112}d_4 = f_{112}d'_4 + f_{112}d_4^{(1,1)} + f_{112}d_4^{(1,2)}$, since $(1,1)$ and $(1,2)$ are the distinct pairs.

Lemma 2.2. *Let x represent a face of an imbedded graph (H, u, v) with $p_x > 0$ u-corners. There are $p_x(p_x + 1)$ ways to insert two edge-ends into the u-corners of this face.*

Proof. Since there are p_x u-corners, there are p_x choices for the location of the first edge-end. After the first edge-end is inserted, the number of u-corners is $p_x + 1$. Thus, there are $p_x + 1$ choices for the second edge-end. Hence, there are a total of $p_x(p_x + 1)$ choices (see Figure 2). □

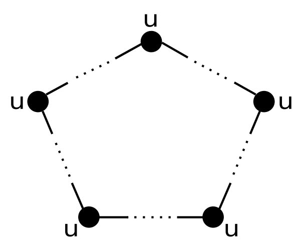

Figure 2: Since $p_x = 5$, there are $30 = 5 * 6$ ways to insert two edge-ends into the u-corners of this face.

Lemma 2.3. *Let x and y be two faces of an imbedded graph (H, u, v), with $p_x > 0$ and $p_y > 0$ u-corners, respectively. There are $2p_xp_y$ ways to insert two edge-ends at root u, such that one edge-end is in face x and the other in face y.*

Proof. There are p_x choices for the edge-end that is inserted into face x, and for each such choice, there are p_y choices for the other edge-end (see Figure 3). Since either of the two edge-ends can be the one that is inserted into face x, we need to multiply p_xp_y by 2. □

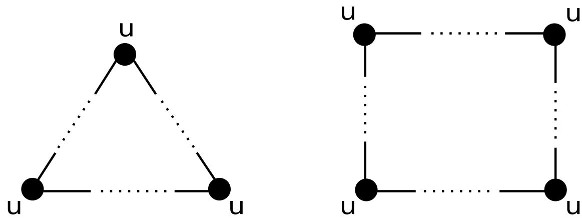

Figure 3: Since $p_x = 3$ and $p_y = 4$, there are $24 = 2 \cdot 3 \cdot 4$ ways to insert two edge-ends with one edge-end in each of the two faces.

Theorem 2.4. *Let $p_1 p_2 \cdots p_r$ be a partition of an integer $n \geq 2$. Suppose that a type-d_i imbedding of a single-rooted graph (G, t) is amalgamated to a type-$f_{p_1 p_2 \cdots p_r} d_j$ imbedding of a double-rooted graph (H, u, v), with deg(v) = deg$(t) = 2$ and deg$(u) = n$. Then the following production holds:*

$$d_i * f_{p_1 p_2 \cdots p_r} d'_j \longrightarrow \left(\sum_{x=1}^{r} p_x(p_x + 1) \right) d_{i+j} + \left(\sum_{x=1}^{r} \sum_{y=x+1}^{r} 2 p_x p_y \right) d_{i+j+1} \tag{2.1}$$

Proof. Since at most one of the r faces incident at root u of H is incident at root v of H, it follows that no matter how the root t of G is amalgamated to u, at most one of the two faces incident at v are affected by this amalgamation. It follows that in the amalgamated graph the two occurrences of v remain on two different faces. There are two cases:
case i. Suppose that both edge-ends incident at root t of graph G are placed into one of the r faces of graph H incident at u. Then no new handle is needed, and thus, the genus increment is 0. The coefficient $\sum_{x=1}^{r} p_x(p_x + 1)$ of d_{i+j} counts the number of ways this can happen. The summation goes from 1 to r, since we can put the two edge-ends incident at t into any of the r faces. The term $p_x(p_x + 1)$ follows from Lemma 2.2.
case ii. Suppose that the two edge-ends incident at root t of graph G are placed into two different faces incident at u. This would necessitate adding a handle — resulting in a genus increment of 1. The coefficient $\sum_{x=1}^{r} \sum_{y=x+1}^{r} 2 p_x p_y$ of d_{i+j+1} counts the number of ways this can happen, by Lemma 2.3. □

Theorem 2.5. *Let $p_1 p_2 \cdots p_r$ be a partition of an integer $n \geq 2$, and let (p_l, p_m) be a pair such that $1 \leq l < m \leq r$. Suppose that a type-d_i imbedding of a single-rooted graph (G, t) is amalgamated to a type-$f_{p_1 p_2 \cdots p_r} d_j^{(p_l, p_m)}$ imbedding of a double-rooted graph (H, u, v), with deg(v) = deg$(t) = 2$ and deg$(u) = n$. Then the following production holds:*

$$d_i * f_{p_1 p_2 \cdots p_r} d_j^{(p_l, p_m)} \longrightarrow \left(\sum_{x=1}^{r} p_x(p_x + 1) \right) d_{i+j} + \left(\left(\sum_{x=1}^{r} \sum_{y=x+1}^{r} 2 p_x p_y \right) - 2 p_l p_m \right) d_{i+j+1} + 2 p_l p_m s_{i+j+1} \tag{2.2}$$

Proof. Let φ_l and φ_m be the two faces incident at root u that are also incident at v, with u occurring p_l times on fb-walk of face φ_l, and p_m times on fb-walk of face φ_m. We note that unless we place one edge-end incident at root t of graph (G,t) into face φ_l and the other edge-end into face φ_m, at most one of the two faces φ_l and φ_m is affected by this amalgamation. Thus, **case i** remains the same as in Theorem 2.4. The first term of the Production (2.2) reflects this similarity. Moreover, **case ii** remains the same as in Theorem 2.4, unless x and y correspond to the faces φ_l and φ_m, which is why we subtract $2p_lp_m$ from the second sum in Production (2.2). If x and y correspond to the faces φ_l and φ_m, then as a result of the amalgamation, the two faces (φ_l and φ_m) combine to become one face having both occurrences of v in its boundary (see Figure 4). The third term of the production reflects this. □

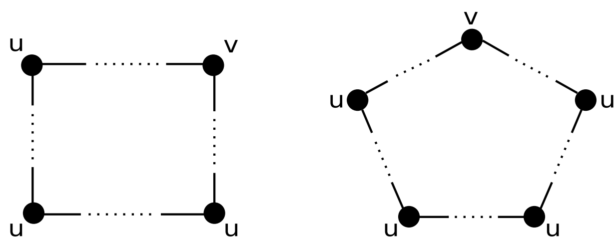

Figure 4: Here $p_l = 3$ and $p_m = 4$. Amalgamation combines the two faces, and the resultant face contains both occurrences of v.

Notation. We sometimes use the shorthand $f_{p_1p_2\cdots p_r}d_j^\bullet$ in place of $f_{p_1p_2\cdots p_r}d_j$, to emphasize the absence of any superscript after d_j.

Theorem 2.6. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$. Suppose that a type-s_i imbedding of a single-rooted graph (G,t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}d_j^\bullet$ imbedding of a double-rooted graph (H,u,v), with $deg(v) = deg(t) = 2$ and $deg(u) = n$. Then the following production holds:*

$$s_i * f_{p_1p_2\cdots p_r}d_j^\bullet \quad \longrightarrow \quad (n^2+n)d_{i+j} \tag{2.3}$$

where the coefficient $n^2 + n = \binom{n+1}{n}$ follows from Proposition 1.2.

Proof. Suppose that in a type-$f_{p_1p_2\cdots p_r}d_j^\bullet$ imbedding of graph (H,u,v), the two occurrences of root-vertex v lie on on two different fb-walks W_1 and W_2 that may or may not contain the root-vertex u. Suppose further that the two occurrences of root-vertex t of graph (G,t) lie on fb-walk X. The two occurrences of root-vertex v continue being on two different fb-walks after the operation of vertex amalgamation, unless the fb-walks W_1 and W_2 combine with the fb-walk X under amalgamation into a single fb-walk. But this cannot happen when the imbedding of (G,t) is a type-s_i imbedding, since a reduction of two faces forces the Euler characteristic to be of odd parity, which is not possible. Thus, there is no genus-increment and all n^2+n resulting imbeddings are type-d_{i+j} imbeddings. □

Sub-partials of $f_{p_1p_2\cdots p_r}s_i$

To define the sub-partials of $f_{p_1p_2\cdots p_r}s_i$ we need the concept of *strands*, which was introduced and used extensively in [13]. When two imbeddings are amalgamated, these strands *recombine* with other strands to form new fb-walks.

Definition 2.7. We define a *u-strand* of an fb-walk of a rooted graph (H, u, v) to be a subwalk that starts and ends with the root vertex u, such that u does not appear in the interior of the subwalk.

The following two types of numbers are the relevant sub-partials of the partial $f_{p_1p_2\cdots p_r}s_i$ for graph (H, u, v):

$f_{p_1p_2\cdots p_r}s'_i$	=	the number of type-$f_{p_1p_2\cdots p_r}s_i$ imbeddings of H such that the two occurrences of v lie in at most one u-strand.
$f_{p_1p_2\cdots p_r}s_i^{(p_l,c)}$	=	the number of type-$f_{p_1p_2\cdots p_r}s_i$ imbeddings of H such that the two occurrences of v lie in two different u-strands of the fb-walk that is represented by p_l, and such that there are $q \geq 1$ intermediate u-corners between the two occurrences of v. We take c to be equal to *min*$(q,\, p_l - q)$, i.e., equal to the smaller number of intermediate u-corners between the two occurrences of root-vertex v.

Note that the last sub-partial would be the same for any other pair (p_a, c) such that $p_a = p_l$.

Theorem 2.8. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$. Suppose that a type-d_i imbedding of a single-rooted graph (G, t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}s'_j$ imbedding of a double-rooted graph (H, u, v), with $deg(v) = deg(t) = 2$ and $deg(u) = n$. Then the following production holds:*

$$d_i * f_{p_1p_2\cdots p_r}s'_j \longrightarrow \left(\sum_{x=1}^{r} p_x(p_x+1)\right) s_{i+j} + \left(\sum_{x=1}^{r}\sum_{y=x+1}^{r} 2p_xp_y\right) s_{i+j+1} \tag{2.4}$$

Proof. Since both occurrences of root v of H lie in at most one u-strand of one of the r fb-walks, it follows that regardless of how the u-strands *recombine* in the amalgamation process, these two occurrences remain on that same u-strand; thus, in all of the resultant imbeddings, the two occurrences of v are on the same fb-walk. As discussed in the proof of Theorem 2.4, there are $\sum_{x=1}^{r} p_x(p_x+1)$ imbeddings that do not result in any genus-increment (corresponding to both edge-ends at t being inserted into the same face at u), whereas there are $\sum_{y=x+1}^{r} 2p_xp_y$ imbeddings that result in a genus increment of 1 (corresponding to inserting both edge-ends at t into the different faces at u). □

Theorem 2.9. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$. Then for each distinct p_l, with $l \in \{1, \cdots, r\}$, and for each integer c in the integer interval $[1, \lfloor \frac{p_l}{2} \rfloor]$, when a* *type-$d_i$ imbedding of a single-rooted graph (G,t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}s_j^{(p_l,c)}$ imbedding of a double-rooted graph (H,u,v), with $deg(v) = deg(t) = 2$ and $deg(u) = n$, the following production holds:*

$$\begin{aligned} d_i * f_{p_1p_2\cdots p_r}s_j^{(p_l,c)} \quad \longrightarrow \quad & \left(\left(\sum_{x=1}^{r} p_x(p_x+1)\right) - p_l(p_l+1)\right) s_{i+j} \\ + \quad & 2c(p_l-c)\, d_{i+j} \\ + \quad & [c(c+1)+(p_l-c)(p_l-c+1)]\, s_{i+j} \\ + \quad & \left(\sum_{x=1}^{r}\sum_{y=x+1}^{r} 2p_xp_y\right) s_{i+j+1} \end{aligned} \tag{2.5}$$

Proof. Let φ_l be the face corresponding to p_l, and let w_1 and w_2 be the two (different) u-strands that contain the two occurrences of root v of H (with c intermediate u-corners between the two occurrences of v). It follows that unless the two edge-ends incident at root t of G are both placed into the face φ_l, the two occurrences of root v will lie on the same fb-walk after amalgamation. The first and last terms of the production reflect this.

Now we consider the case when the two edge-ends incident at root t of graph (G,t) are both placed into the face φ_l. Let e_{start_1} and e_{start_2} be the initial edge-ends of u-strands w_1 and w_2, similarly let e_{end_1} and e_{end_2} be the terminal edge-ends of u-strands w_1 and w_2 (we consider that a u-strand starts and ends at root u). This is illustrated in Figure 5.

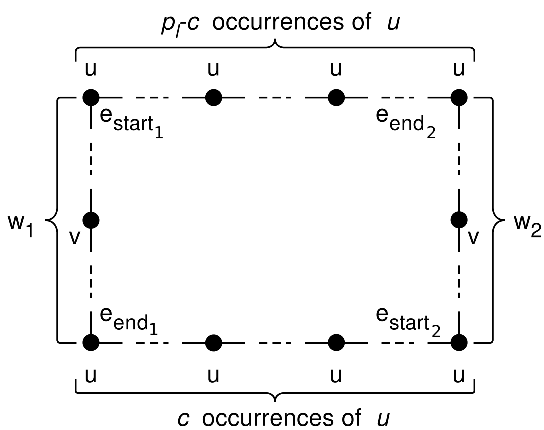

Figure 5: fb-walk of a type-$f_{p_1p_2\cdots p_r}s_j^{(p_l,c)}$ imbedding.

It is clear that in the fb-walk of the face φ_l, these four edge-ends appear in $e_{start_1}, e_{end_1}, e_{start_2}, e_{end_2}$ cyclic order. If one of the two edge-ends incident at root t is placed between e_{end_1} and e_{start_2} and the other between e_{end_2} and e_{start_1}, then after the strands are recombined, one of the u-strands containing one occurrence of root v clearly recombines with the one t-strand of (G,t) to make a new face (see Figure 6, left).

It follows that in this case the two occurrences of root v will lie on two different faces. Since there are a total of p_l u-corners in face φ_l, and there are c intermediate u-corners between the two occurrences of root v of graph (H,u,v), there are $2c(p_l-c)$ ways in all

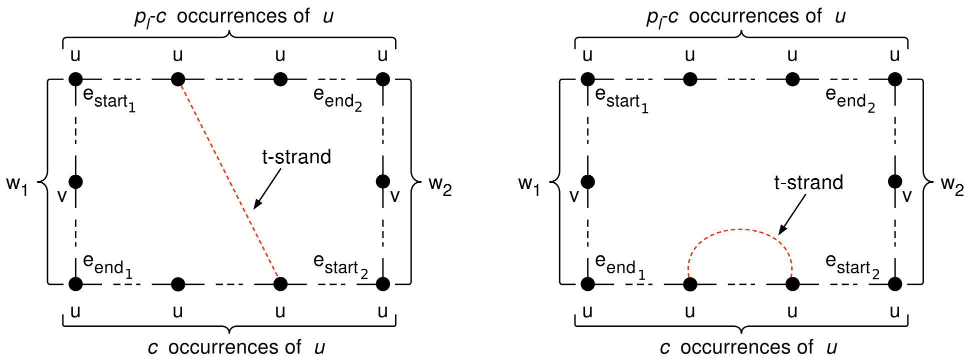

Figure 6: The two ways of inserting t-strands.

of inserting the two edge-ends incident at root t of graph (G,t) in this way. We multiply by 2 since either of the two edge-ends can be chosen as the first edge-end. The second term of the production reflects this case.

If both of the edge-ends incident at root t are placed between e_{end_1} and e_{start_2}, or between e_{end_2} and e_{start_1}, then the two occurrences of root v lie on the same face after u-strands and t-strands are recombined (see Figure 6, right). There are $c(c+1)+(p_l-c)(p_l-c+1)$ ways this can happen, since there are c and $p_l - c$ intermediate u-corners between w_1 and w_2. □

Notation. We sometimes use the shorthand $f_{p_1p_2\cdots p_r}s_j^{\bullet}$ in place of $f_{p_1p_2\cdots p_r}s_j$, to emphasize the absence of any superscript after s_j.

Theorem 2.10. *Let $p_1p_2\cdots p_r$ be a partition of an integer $n \geq 2$. Suppose that a type-s_i imbedding of a single-rooted graph (G,t) is amalgamated to a type-$f_{p_1p_2\cdots p_r}s_j$ imbedding of a double-rooted graph (H,u,v), with $deg(v) = deg(t) = 2$ and $deg(u) = n$. Then the following production holds:*

$$s_i * f_{p_1p_2\cdots p_r}s_j^{\bullet} \quad\longrightarrow\quad (n^2+n)s_{i+j} \tag{2.6}$$

Proof. Since the two occurrences of root v of H lie on the same fb-walk. One necessary condition for the operation of vertex amalgamation to change this is that both edge-ends at root t of G are inserted into that face. However, since both occurrences of root t are on the same fb-walk, both ends of each t-strand lie in the same u-corner of that face, as illustrated in Figure 7. This implies that no new handle is needed as a result of the amalgamation. Thus, there is no genus-increment. □

Corollary 2.11. *Let $(X,v) = (G,t)*(H,u,v)$, where $deg(v) = deg(t) = 2$ and $deg(u) = n$ for $n \geq 2$. Then for*

$$\alpha_{p_1p_2\cdots p_r} = \sum_{x=1}^{r} p_x(p_x+1) \quad \textit{and} \quad \beta_{p_1p_2\cdots p_r} = \sum_{x=1}^{r}\sum_{y=x+1}^{r} 2p_xp_y$$

we have

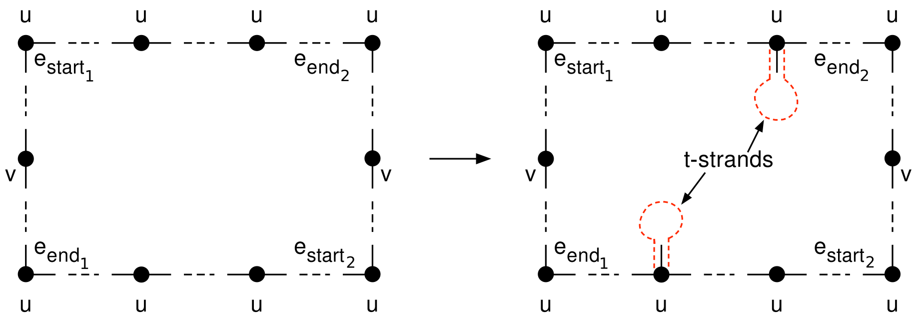

Figure 7: Even after the amalgamation, the two occurrences of v remain on the same fb-walk.

$$
\begin{aligned}
d_k(X) = \sum_{\substack{\text{over all partitions}\\ p_1p_2\cdots p_r \text{ of } n}} \Bigg[& \sum_{i=0}^{k} \alpha_{p_1p_2\cdots p_r}\ d_{k-i} f_{p_1p_2\cdots p_r} d'_i \\
& + \sum_{i=0}^{k-1} \beta_{p_1p_2\cdots p_r}\ d_{k-i-1} f_{p_1p_2\cdots p_r} d'_i \\
& + \sum_{i=0}^{k} \sum_{\substack{\text{over all}\\ \text{distinct } (p_l,p_m)\\ l<m}} \alpha_{p_1p_2\cdots p_r}\ d_{k-i} f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)} \\
& + \sum_{i=0}^{k-1} \sum_{\substack{\text{over all}\\ \text{distinct } (p_l,p_m)\\ l<m}} (\beta_{p_1p_2\cdots p_r} - 2p_lp_m)\ d_{k-i-1} f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)} \\
& + \sum_{i=0}^{k} (n^2+n)\ s_{k-i} f_{p_1p_2\cdots p_r} d_i^{\bullet} \\
& + \sum_{i=0}^{k} \sum_{\substack{\text{over all}\\ \text{distinct } p_l}} \sum_{c=1}^{\lfloor \frac{p_l}{2} \rfloor} 2c(p_l - c)\ d_{k-i} f_{p_1p_2\cdots p_r} s_i^{(p_l,c)} \Bigg]
\end{aligned} \tag{2.7}
$$

Proof. This equation is derived from Theorems 2.4, 2.5, 2.6 and 2.9 by a routine transposition of the productions that have the single-root partial d on their right-hand-side. □

Corollary 2.12. *Let* $(X, v) = (G, t) * (H, u, v)$*, where* $deg(v) = deg(t) = 2$ *and* $deg(u) = n$ *for* $n \geq 2$*. Then for*

$$
\alpha_{p_1p_2\cdots p_r} = \sum_{x=1}^{r} p_x(p_x+1) \quad \textit{and} \quad \beta_{p_1p_2\cdots p_r} = \sum_{x=1}^{r} \sum_{y=x+1}^{r} 2p_xp_y
$$

we have

$$
\begin{aligned}
s_k(X) = \sum_{\substack{\text{over all partitions}\\ p_1p_2\cdots p_r \text{ of } n}} \Bigg[& \sum_{i=0}^{k} \alpha_{p_1p_2\cdots p_r}\; d_{k-i} f_{p_1p_2\cdots p_r} s'_i \\
+ & \sum_{i=0}^{k-1} \beta_{p_1p_2\cdots p_r}\; d_{k-i-1} f_{p_1p_2\cdots p_r} s'_i \\
+ & \sum_{i=0}^{k-1} \sum_{\substack{\text{over all}\\ \text{distinct } (p_l,p_m)\\ l<m}} 2p_lp_m\; d_{k-i-1} f_{p_1p_2\cdots p_r} d_i^{(p_l,p_m)} \\
+ & \sum_{i=0}^{k} \sum_{\substack{\text{over all}\\ \text{distinct } p_l}} \sum_{c=1}^{\lfloor \frac{p_l}{2} \rfloor} \Big(c(c+1) + (p_l-c)(p_l-c+1) \\
& \quad + \alpha_{p_1p_2\cdots p_r} - p_l(p_l+1) \Big)\; d_{k-i} f_{p_1p_2\cdots p_r} s_i^{(p_l,c)} \\
+ & \sum_{i=0}^{k-1} \sum_{\substack{\text{over all}\\ \text{distinct } p_l}} \sum_{c=1}^{\lfloor \frac{p_l}{2} \rfloor} \beta_{p_1p_2\cdots p_r}\; d_{k-i-1} f_{p_1p_2\cdots p_r} s_i^{(p_l,c)} \\
+ & \sum_{i=0}^{k} (n^2+n)\; s_{k-i} f_{p_1p_2\cdots p_r} s_i^{\bullet} \Bigg]
\end{aligned} \tag{2.8}
$$

Proof. This equation is derived from Theorems 2.5, 2.8, 2.9 and 2.10 by a routine transposition of the productions that have the single-root partial s on their right-hand-side. □

Remark 2.13. In writing Recursions 2.7 and 2.8, we have suppressed indication of graphs G and H as arguments, in order that they not occupy too many lines. In the examples to follow, we see how restriction of these recursions to particular genus distributions of interest greatly simplifies them. The reason for placing the index variable i of each sum with the second factor, rather than the first, also becomes clear in the applications.

3 Open chains of copies of K_4

We can specify a sequence of *open chains* of copies of a double-rooted graph (G, u, v) recursively.

$$(X_1, t_1) = (G, v) \qquad \text{(suppressing co-root } u) \tag{3.1}$$

$$(X_m, t_m) = (X_{m-1}, t_{m-1}) * (G, u, v) \;\; \text{for } m \geq 1 \tag{3.2}$$

For example, consider a chain of copies of the graph K_4 with one edge subdivided as in Figure 8. We observe that each of the amalgamations results in a vertex of degree 5.

By face-tracing the imbeddings of K_4, we obtain Table 1.

By using Recurrences (2.7) and (2.8) for $deg(u) = n = 3$, and the values from Table 1,

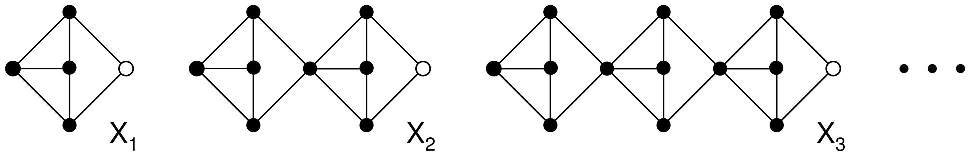

Figure 8: X_m is an open chain of m copies of K_4.

k	$f_{111}d'_k$	$f_{21}d_k^{(2,1)}$	$f_{21}s_k^{(2,1)}$	$f_3d'_k$	d_k	s_k	g_k
0	2	0	0	0	2	0	2
1	0	6	6	2	8	6	14

Table 1: Nonzero partials of (G, u, v).

we obtain the following two recurrences, for $m \geq 2, k \geq 0$:

$$d_k(X_m) = 12d_k(X_{m-1}) + 24s_k(X_{m-1}) + 96d_{k-1}(X_{m-1}) + 96s_{k-1}(X_{m-1}) \quad (3.3)$$

$$s_k(X_m) = 48d_{k-2}(X_{m-1}) + 36d_{k-1}(X_{m-1}) + 72s_{k-1}(X_{m-1}) \quad (3.4)$$

Another way of obtaining these recurrences without having to use Recurrences (2.7) and (2.8), is to first list all productions that are relevant for the example at hand (i.e. corresponding to the non-zero double-root partials) using Theorems 2.4–2.10; we list the productions for this example in Table 2. We can then transpose these productions, and use the values of double-root partials from Table 1 on the transposed productions to come up with the desired recurrences.

$d_i * f_{111}d'_j$	$\longrightarrow\ 6d_{i+j} + 6d_{i+j+1}$
$s_i * f_{111}d^\bullet_j$	$\longrightarrow\ 12d_{i+j}$
$d_i * f_{21}d_j^{(2,1)}$	$\longrightarrow\ 8d_{i+j} + 4s_{i+j+1}$
$d_i * f_{21}s_j^{(2,1)}$	$\longrightarrow\ 2d_{i+j} + 6s_{i+j} + 4s_{i+j+1}$
$s_i * f_{21}d^\bullet_j$	$\longrightarrow\ 12d_{i+j}$
$s_i * f_{21}s^\bullet_j$	$\longrightarrow\ 12s_{i+j}$
$d_i * f_3d'_j$	$\longrightarrow\ 12d_{i+j}$
$s_i * f_3d^\bullet_j$	$\longrightarrow\ 12d_{i+j}$

Table 2: The non-zero productions when $deg(u) = 3$.

Using these recurrences and the values of single-root partials in Table 1, we obtain the values of single-root partials for X_2, that are listed in Table 3. We can then use values of the partials for X_2 to obtain the values of single-root partials for X_3, also listed in Table 3. We can iterate this to obtain the genus distribution of X_m for any value of m.

	X_2			X_3		
k	d_k	s_k	g_k	d_k	s_k	g_k
0	24	0	24	288	0	288
1	432	72	504	9216	864	10080
2	1344	816	2160	84096	21888	105984
3	0	384	384	216576	127872	344448
4				36864	92160	129024

Table 3: Single-root partials of X_2 and X_3.

4 Another example

As another illustration of the method, we compute the recurrences for the open chains of a graph (G, s, t) in which $deg(u) = n = 6$ (see Figure 9). Where, as in previous example, X_1 is the graph G with root s suppressed.

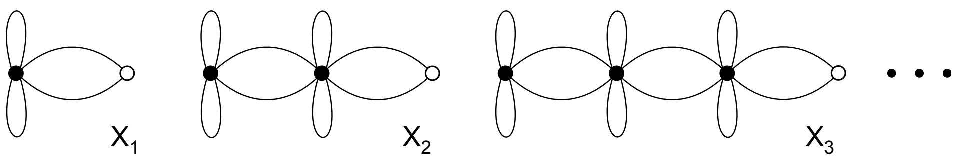

Figure 9: X_m is an open chain of m copies of G.

By face-tracing the imbeddings of (G, u, v), we obtain Table 4.

type	$k = 0$	$k = 1$
$f_{51}d^{(5,1)}$	0	16
$f_{2211}d^{(2,2)}$	8	0
$f_{2211}d^{(2,1)}$	16	0
$f_{42}s^{(4,2)}$	0	8
$f_{42}d^{(4,2)}$	0	16
$f_{33}d^{(3,3)}$	0	8
$f_{3111}d^{(3,1)}$	16	0
$f_{51}s^{(5,2)}$	0	32
d_k	40	40
s_k	0	40
g_k	40	80

Table 4: Nonzero partials of (G, u, v).

Using Recurrences (2.7) and (2.8), we obtain the following two recurrences for $m \geq$

$2, k \geq 0$:

$$d_k(X_m) = 672 d_k(X_{m-1}) + 1680 s_k(X_{m-1}) + 2352 d_{k-1}(X_{m-1}) + 1680 s_{k-1}(X_{m-1}) \quad (4.1)$$

$$s_k(X_m) = 1008 d_{k-2}(X_{m-1}) + 1008 d_{k-1}(X_{m-1}) + 1680 s_{k-1}(X_{m-1}) \quad (4.2)$$

Table 5 records the values that these recurrences give us for X_2 and X_3.

	X_2			X_3		
k	d_k	s_k	g_k	d_k	s_k	g_k
0	26880	0	26880	18063360	0	18063360
1	188160	40320	228480	257402880	27095040	284497920
2	161280	147840	309120	867041280	284497920	1151539200
3	0	40320	40320	695439360	600606720	1296046080
4				67737600	230307840	298045440

Table 5: Single-root partials of X_2 and X_3.

5 Conclusions

Results in this paper enable us to compute genus distributions of open chains in which the degree of the amalgamated vertex can be arbitrarily large, a significant improvement from the previous results where it was limited to being 4-valent.

Using the methods developed in this paper, one can compute the following:

- the genus distribution of the vertex-amalgamation of two single-rooted graphs (G, t) and (H, u), when $deg(t) = 2$ and $deg(u)$ is arbitrarily large. One simply adds up all terms on the right-hand-sides of Recurrences (2.7) and (2.8) after converting double-root partials to single-root partials (by ignoring the second root).
- recurrences for the genus distribution of the sequence of open chains of double-rooted graphs (H, u, v), where $deg(v) = 2$ and $deg(u)$ can be arbitrarily large, provided that the genus distribution of (H, u, v) is known and is further analyzed into a partitioned genus distribution.

It is interesting to note that extending the methods developed in this paper to amalgamating a single-rooted graph (G, t) with a double-rooted graph (H, u, v), with $deg(t) \geq 3$ and $deg(v) \geq 3$, might not be so straight-forward. As illustrated in Example 5.1, the genus-increment may sometimes be negative (see Research Problem 1 below).

Example 5.1. Figure 10 shows two toroidal imbeddings $\iota_1 : (D_3, u) \to S_1$ and $\iota_2 : (D_3, u) \to S_1$ of the single-rooted dipole (D_3, u).

One of the 40 imbeddings of the amalgamated graph $X = (D_3, u) * (D_3, u)$ that is consistent with those two imbeddings is also shown in the figure. Note that this is also a toroidal imbedding, since

$$V - E + F = 3 - 6 + 3 = 0.$$

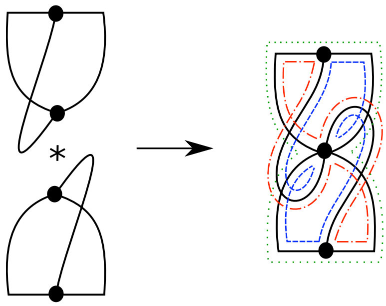

Figure 10: A consistent imbedding of $D_3 * D_3$ with negative genus-increment.

Thus, the genus-increment in this case is -1.

Research Problem 1. Develop methods for computing the genus distributions when both amalgamated vertices have abitrarily large degrees. For instance, one might augment the present approach with other surgical operations, such as splitting a vertex.

Research Problem 2. Develop methods to solve simultaneous recurrences like (3.3), (3.4) and (4.1), (4.2).

Research Problem 3. As noted in [26], the numbers such as the ones computed in Table 1 and Table 3 appear to support the conjecture that all graphs have unimodal genus distributions. A natural question to ask is whether the vertex-amalgamation of two graphs with unimodal distributions has a unimodal genus distribution.

Research Problem 4. Results of [26] have been successfully used to compute the genus distributions of cubic outerplanar graphs ([10]). Can the results of this paper be similarly used to compute the genus distributions of other non-linear families of graph?

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